WISCONSIN STANDARDS FOR

Mathematics

Wisconsin Department of Public Instruction

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Wisconsin Department of Public Instruction
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Foreword

In May 10, 2021, I formally adopted the Wisconsin Standards for Mathematics. This revised set of academic standards provides a foundational framework identifying the knowledge and skills in mathematics Wisconsin students should learn at different grade levels or bands of grades.

The standards are a result of a concerted effort led by Wisconsin educators and stakeholders who shared their expertise in mathematics and teaching from kindergarten through higher education. Feedback was provided by the public and the Wisconsin Legislature for the writing committee to consider as part of Wisconsin's process for reviewing and revising academic standards.

Mathematics is an essential part of a comprehensive PK-12 education for all students. Wisconsin students learn to use mathematics to understand and empower themselves and their worlds. The knowledge, skills, and habits of mind gained through mathematics education in Wisconsin schools support the Wisconsin Department of Public instruction's vision of helping all students graduate college and career ready.

Wisconsin's 2021 standards for mathematics focus on ensuring every student has the ability to develop deep mathematical understanding as a confident and capable learner. To develop deep mathematical understanding, positive mathematical identity, as well as strong mathematical agency, students need instruction that recognizes the broader purpose of mathematics. To this end, the Wisconsin Standards for Mathematics result in the following:

- Wisconsin's students will develop deep mathematics understanding, so that they may experience joy and confidence in themselves as mathematicians.
- Wisconsin's students will develop as mathematicians through both mathematical practices and content.
- Wisconsin's students will be flexible users of mathematics as they use mathematics to understand the world and question and critique the world using mathematical justifications.
- Wisconsin's students will have expanded professional opportunities in a wide variety of careers.

The knowledge and skills described in this revised set of standards provide a framework with actionable indicators for mathematics classroom experiences. The Wisconsin Department of Public Instruction will continue to build on this work to support implementation of the standards with resources for the field.

I am excited to share the Wisconsin Standards for Mathematics, which aims to build skills, knowledge, and engagement opportunities for all Wisconsin students.

Carolyn Stanford Taylor
State Superintendent

Acknowledgements

The Wisconsin Department of Public Instruction (DPI) wishes to acknowledge the ongoing work, commitment, and various contributions of individuals to revise our state's academic standards for mathematics. Thank you to the State Superintendent's Standards Review Council for their work and guidance through the standards process. A special thanks to the Mathematics Writing Committee for taking on this important project that will shape the classrooms of today and tomorrow. Thanks to the many staff members across the division and other teams at DPI who have contributed their time and talent to this project. Finally, a special thanks to Wisconsin educators, businesspeople, parents, and citizens who provided comment and feedback to drafts of these standards.

Wisconsin Standards for Mathematics

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Section I

Wisconsin's Approach to Academic Standards

Purpose of the Document

The purpose of this guide is to improve mathematics education for students and for communities. The Wisconsin Department of Public Instruction (DPI) has developed standards to assist Wisconsin educators and stakeholders in understanding, developing and implementing course offerings and curriculum in school districts across Wisconsin.

This publication provides a vision for student success and follows *The Guiding Principles for Teaching and Learning (2011)*. In brief, the principles are:

1. Every student has the right to learn.
2. Instruction must be rigorous and relevant.
3. Purposeful assessment drives instruction and affects learning.
4. Learning is a collaborative responsibility.
5. Students bring strengths and experiences to learning.
6. Responsive environments engage learners.

Program leaders will find the guide valuable for making decisions about:

- Program structure and integration
- Curriculum redesign
- Staffing and staff development
- Scheduling and student grouping
- Facility organization
- Learning spaces and materials development
- Resource allocation and accountability
- Collaborative work with other units of the school, district, and community

What Are the Academic Standards?

Wisconsin Academic Standards specify what students should know and be able to do in the classroom. They serve as goals for teaching and learning. Setting high standards enables students, parents, educators, and citizens to know what students should have learned at a given point in time. In Wisconsin, all state standards serve as a model. Locally elected school boards adopt academic standards in each subject area to best serve their local communities. We must ensure that all children have equal access to high-quality education programs. Clear statements about what students must know and be able to do are essential in making sure our schools offer opportunities to get the knowledge and skills necessary for success beyond the classroom.

Adopting these standards is voluntary. Districts may use the academic standards as guides for developing local grade-by-grade level curriculum. Implementing standards may require some school districts to upgrade school and district curriculums. This may result in changes in instructional methods and materials, local assessments, and professional development opportunities for the teaching and administrative staff.

What is the Difference Between Academic Standards and Curriculum?

Standards are statements about what students should know and be able to do, what they might be asked to do to give evidence of learning, and how well they should be expected to know or do it. Curriculum is the program devised by local school districts used to prepare students to meet standards. It consists of activities and lessons at each grade level, instructional materials, and various instructional techniques. In short, standards define what is to be learned at certain points in time, and from a broad perspective, what performances will be accepted as evidence that the learning has occurred. Curriculum specifies the details of the day-to-day schooling at the local level.

Developing the Academic Standards

DPI has a transparent and comprehensive process for reviewing and revising academic standards. The process begins with a notice of intent to review an academic area with a public comment period. The State Superintendent's Standards Review Council examines those comments and may recommend revision or development of standards in that academic area. The state superintendent authorizes whether or not to pursue a revision or development process. Following this, a state writing committee is formed to work on those standards for all grade levels. That draft is then made available for open review to get feedback from the public, key stakeholders, educators, and the Legislature with further review by the State Superintendent's Standards Review Council. The state superintendent then determines adoption of the standards.

Aligning for Student Success

To build and sustain schools that support every student in achieving success, educators must work together with families, community members, and business partners to connect the most promising practices in the most meaningful contexts. The release of the *Wisconsin Standards for Mathematics* provides a set of important academic standards for school districts to implement. This is connected to a larger vision of every child graduating college and career ready. Academic standards work together with other critical principles and efforts to educate every child to graduate college and career ready. Here, the vision and set of Guiding Principles form the foundation for building a supportive process for teaching and learning rigorous and relevant content. The following sections articulate this integrated approach to increasing student success in Wisconsin schools and communities.

Relating the Academic Standards to All Students

Grade-level standards serve as goals for teaching and learning. Relating the standards to students requires all educators to center the learner as they plan for instruction and assessment and create systems that prioritize student understanding. This means that ALL students need to have access to grade-level high quality instruction and assessment in ways that fit their strengths, needs, and interests. This applies to students with Individualized Education Plans (IEPs), emerging bilingual learners, and gifted and talented pupils, consistent with all other students.

Academic standards serve as a valuable basis for establishing concrete, meaningful goals as part of each student's developmental progress and demonstration of proficiency. Students with IEPs must be provided specially designed instruction that meets their individual needs. It is expected that each individual student with an IEP will require unique services and supports matched to their strengths and needs in order to close achievement gaps in grade-level standards. Alternate standards are only available for students with the most significant cognitive disabilities. Multilingual learners deserve high expectations that emphasize the vital role of language and communication in solving mathematical problems, in developing mathematical thinking, and demonstrating knowledge in classroom interactions (Chval, Pinnow, Smith, and Trigos-Carrillo 2021, xv). As with all students, gifted and talented students should experience daily engagement with the Standards for Mathematical Practice. Students need ongoing opportunities to experience the joy of investigating rich concepts in depth and applying mathematical reasoning and justification to a variety of scientific, engineering, and other problems. Pacing for gifted and talented students means that they have the time and opportunity to delve deeply and creatively into topics, projects, and problems of interest (Johnsen and Sheffield 2013, 15-16, 18-19).

Our Vision: Every Child a Graduate, College and Career Ready

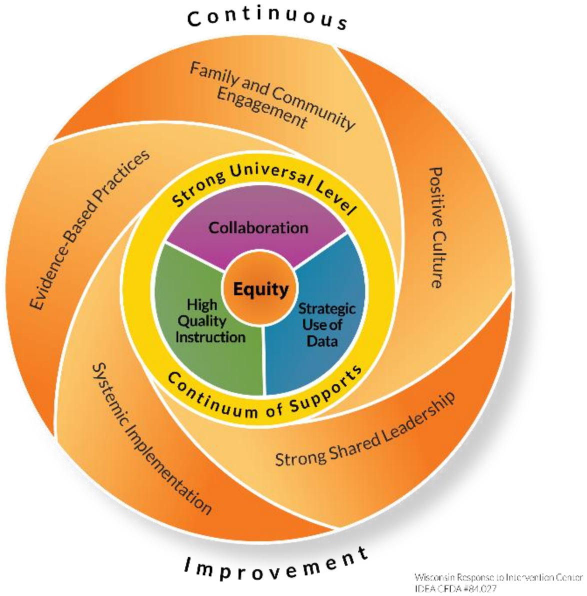
We are committed to ensuring every child graduates from high school academically prepared and socially and emotionally competent. A successful Wisconsin student is proficient in academic content and can apply their knowledge through skills such as critical thinking, communication, collaboration, and creativity. The successful student will also possess critical habits such as perseverance, responsibility, adaptability, and leadership. This vision for every child as a college and career ready graduate guides our beliefs and approaches to education in Wisconsin.

Guided by Principles

All educational initiatives are guided and impacted by important and often unstated attitudes or principles for teaching and learning. *The Guiding Principles for Teaching and Learning (2011)* emerge from research and provide the touchstone for practices that truly affect the vision of Every Child a Graduate Prepared for College and Career. When made transparent, these principles inform what happens in the classroom, direct the implementation and evaluation of programs, and most importantly, remind us of our own beliefs and expectations for students.

Ensuring a Process for Student Success

For Wisconsin schools and districts, implementing the *Framework for Equitable Multi-Level Systems of Supports (2017)* means providing equitable services, practices, and resources to every learner based upon responsiveness to effective instruction and intervention. In this system, high-quality instruction, strategic use of data, and collaboration interact within a continuum of supports to facilitate learner success. Schools provide varying types of supports with differing levels of intensity to proactively and responsibly adjust to the needs of the whole child. These include the knowledge, skills, and habits learners need for success beyond high school, including developmental, academic, behavioral, social, and emotional skills.



Connecting to Content: Wisconsin Academic Standards

Within this vision for increased student success, rigorous, internationally benchmarked academic standards provide the content for high-quality curriculum and instruction and for a strategic assessment system aligned to those standards. With the adoption of the standards, Wisconsin has the tools to design curriculum, instruction, and assessments to maximize student learning. The standards articulate what we teach so that educators can focus on how instruction can best meet the needs of each student. When implemented within an equitable multi-level system of support, the standards can help to ensure that every child will graduate college and career ready.

Section II

Wisconsin Standards for Mathematics

What is Mathematics Education in Wisconsin?

Wisconsin's Vision for Mathematics

The Wisconsin vision for mathematics is shaped by Wisconsin practitioners, experts, and the business community, and is informed by work at the national level and in other states. The overarching goal of Wisconsin's vision for mathematics is for students to see themselves as confident doers and learners of mathematics, supporting the department's vision to be college and career ready.

1. Wisconsin's students will develop deep mathematics understanding so that they may experience joy and confidence in themselves as mathematicians.
2. Wisconsin's students will develop as mathematicians through both mathematical practices and content.
3. Wisconsin's students will be flexible users of mathematics as they use mathematics to understand the world and also question and critique the world using mathematical justifications.
4. Wisconsin's students will have expanded professional opportunities in a wide variety of careers.

Wisconsin's Guiding Principles for Teaching and Learning (2011) provide important guidance for approaching the vision of mathematics. Each of the six guiding principles has implications for equity, pedagogy, instruction, and assessment. Mathematics educators should consider how teaching and learning systems and structures are in service of students with respect to each of the principles.

Every student has the right to learn significant mathematics.

Mathematical proficiency is essential for every student in Wisconsin. Students need to be able to formulate, represent, and solve problems; explain and justify solutions and solution paths; and see mathematics as sensible, useful, and worthwhile. In order to achieve this vision, all students must have access to challenging, rigorous, and meaningful mathematics. Schools and classrooms need to be organized to convey the message that all students can learn mathematics and should be expected to achieve.

Mathematics instruction should be rigorous and relevant.

Teachers focus on engaging students in using mathematical reasoning, making mathematical connections, and modeling and representing mathematical ideas in a variety of ways. The mathematics curriculum needs to integrate and sequence important mathematical ideas so that mathematics makes sense. Teachers use rich tasks to engage students in the development of conceptual understanding and procedural skills. An emphasis on connections within mathematics helps students see mathematics as a coherent and integrated whole rather than as a set of isolated and disconnected skills and procedures. Through mathematical applications, students recognize the usefulness of mathematics and appreciate the need to study and understand mathematical skills and concepts.

Purposeful assessment drives mathematics instruction and affects learning.

Teachers measure mathematical proficiency by using a variety of purposeful assessments before, during, and after instruction. Rich assessment tasks ask students to demonstrate their understanding by representing mathematical situations, solving problems as developed in the classroom, and justifying their solutions. Valuable assessments provide both students and teachers with the opportunity to reflect on students' mathematical communication, precision, and reasoning. Teachers use resulting data to adapt their instruction and the learning environment so that all students will understand new mathematics concepts and content.

Learning mathematics is a collaborative responsibility.

Collaborative structures, within the mathematics classroom as well as in the school community, support the teaching and learning of mathematics. Students develop mathematical habits of mind through purposeful interactions in the classroom. Teachers co-create contexts, conditions, and assessment strategies for an interdependent learning environment. Opportunities for students to communicate the solutions, solution paths, and justifications are present in mathematics lessons.

Students bring strengths and experiences to mathematics learning.

Students bring informal experiences of mathematics from their home and community to the mathematics classroom. They may enter classrooms with varying levels of mathematical misconceptions and confidence in their ability to do mathematics. Schools and teachers must build upon students' prior knowledge and intuitive understanding of mathematical ideas in order to connect the formal study of mathematics to students' ongoing experiences. Teachers need to continually identify students' strengths and weaknesses as a basis to develop tasks and experiences that will capitalize on student strengths and address weaknesses and misconceptions.

Responsive environments engage mathematics learners.

Teachers utilize strategies that create effective mathematics environments. These environments use high quality mathematics curriculum and instruction in response to the understanding that not all students learn at the same pace or in the same way. Student engagement, perseverance, and learning are increased when teachers respond to students' interests, learning profiles, and readiness. The Standards for Mathematical Practice are evident in a responsive environment.

Efforts to create and sustain a district or school mathematics program that effectively implements the *Wisconsin Standards for Mathematics* should involve Wisconsin's Guiding Principles for Teaching and Learning (2011). This must be ongoing work in all Wisconsin schools and districts. It is critical that these Guiding Principles are used as a framework to continually inform the conversations around how to best create systems and structures that are designed for equitable outcomes. These conversations include, but are not limited to, determining a district's vision for mathematics and considering how pedagogy impacts instruction and assessment.

Wisconsin's Approach to Academic Standards for Mathematics

The *Wisconsin Standards for Mathematics* (2021) are built on the foundation of existing standards (National Governors Association Center for Best Practices, Council of Chief State School Officers 2010) and incorporate shifts that reflect new research and broader expectations of mathematics. Three of the five shifts are from the *Wisconsin Standards for Mathematics (2010)*, but have been expanded upon to emphasize advancing educational equity in mathematics. Two of the five shifts are new and unique to Wisconsin. There are five important shifts from previous standards (2010) to these revised *Wisconsin Standards for Mathematics (2021)*. Identifying the key shifts builds understanding of how these standards differ from previous standards. The shifts also serve to guide educators in identifying what is necessary in standards-aligned instruction and assessment.

Shift #1: Learning mathematics emphasizes recognizing, valuing, and fostering mathematical identities and agency in all students.

The *Wisconsin Standards for Mathematics* (2021) expects opportunities for inclusion of broader ways to think and do mathematics. This shift supports recognizing and valuing the mathematical ways of thinking students bring with them to school, mathematics from their culture, their families, or previous grade level. By leveraging multiple mathematical competencies, drawing on multiple resources of knowledge, and going deep into the mathematical concepts, students develop stronger mathematical understanding (Aguirre, Martin, and Mayfield-Ingram 2013, 43).

Shift #2: All students are flexible users of mathematics who see how mathematics can be used to understand their world and the world around them.

The *Wisconsin Standards for Mathematics* (2021) call for empowering students to be thinkers and doers of mathematics. Engaging students in mathematizing and modeling is a way to bring this shift to life in students' mathematical journeys. The standards are calling for an intentional pairing of the Standards for Mathematical Practice and the Standards for Mathematical Content that allow for students to be mathematically curious and gain a lifelong appreciation of mathematics and how mathematics is used to understand, critique, and create solutions for the world (NCTM 2020, 15).

Shift #3: All students engage in mathematics that is focused on developing deep understanding of and connections among mathematical concepts in order to gain strong foundations to move their mathematics learning forward.

The *Wisconsin Standards for Mathematics* (2021) continue to call for a focus in mathematics content at each grade level. The standards significantly narrow and deepen the way time and energy are spent. This means focusing deeply on the major work of each grade as follows:

- In grades K–2: Concepts, skills, and problem solving related to addition and subtraction
- In grades 3–5: Concepts, skills, and problem solving related to multiplication and division of whole numbers and fractions
- In grade 6: Ratios and proportional relationships, and early algebraic expressions and equations
- In grade 7: Ratios and proportional relationships, and arithmetic of rational numbers
- In grade 8: Linear algebra and linear functions

The *Wisconsin Standards for Mathematics* (2021) focus deeply on the major work of each grade. This focus will help students gain strong foundations, including a solid understanding of concepts to support their mathematical experiences. Students develop a strong foundational knowledge and deep conceptual understanding and are able to transfer mathematical skills and understanding across concepts and grades (National Mathematics Advisory Panel 2008, 15-20).

Shift #4: All students engage in coherent mathematics that connects concepts and mathematical thinking within and across domains and grades.

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts. Therefore, the standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years. Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics.

Shift #5: All students engage in rigorous mathematics within a balanced approach developing conceptual understanding, procedural flexibility and efficiency and application to authentic contexts.

Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, flexible and efficient procedural skills, and application.

Conceptual understanding: The standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.

Flexible and efficient procedural skills: The standards call for efficiency and accuracy in calculation. Flexible and efficient procedural skills build from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems (NCTM 2014).

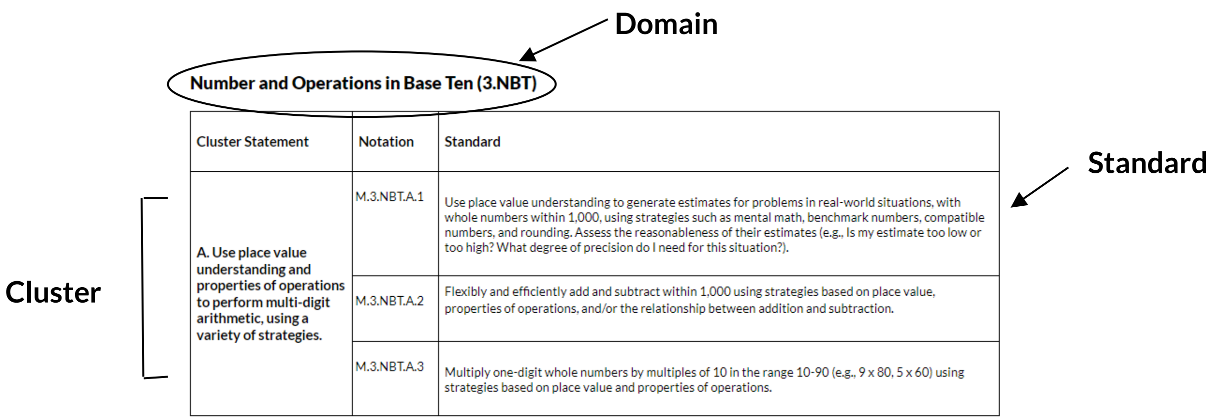
Application: The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural flexibility and efficiency.

The *Wisconsin Standards for Mathematics* (2021) may be taught and integrated through a variety of classes and experiences. Each district, school, and program area should determine the means by which students meet these standards. Through the collaboration of multiple stakeholders, these foundational standards will set the stage for high-quality, successful, contemporary mathematics courses and programs throughout Wisconsin's PK-12 systems.

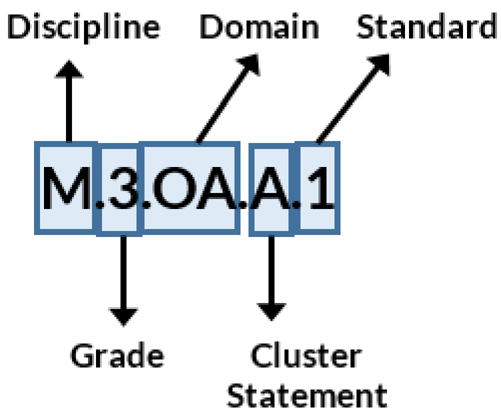
Content Standards Structure

Wisconsin Standards for Mathematics have the following design features:

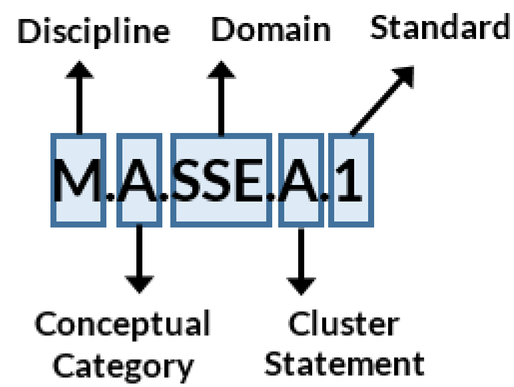
- **Domains**—Larger groups of related standards. Standards from different domains may sometimes be closely related. In high school standards there is an additional grouping of domains called **Conceptual Categories**.
- **Clusters**—Groups of related standards. Notice that the cluster statements appear at the far left of each standard table to visually highlight an emphasis on the cluster statement. Individual standards stem from cluster statements and provide the details of what students should understand and be able to do. Standards from different clusters may sometimes be closely related, because mathematics is a connected subject.
- **Standards**—Define what students should understand and be able to do.
    - **New Standard Numbers**—When standard numbers were changed from those used in 2010 due to the addition of a new standard within an existing cluster, the new standard code appears (e.g., M.K.CC.C.7) as well as the previous code. (e.g., [WI.2010.K.CC.C.6] for reference).
    - **(M) Mathematical Modeling**—Mathematical modeling is best interpreted, not as a collection of isolated topics, but rather in relation to other standards. To support these relationships, content standards that may be particularly valuable in middle school and high school have been indicated with an (M) symbol. The (M) symbol appears following those cluster statements throughout grades 6-12.
    - **(F2Y) First Two Years**—This denotes content standards that should be completed by all students in their first two years of high school mathematics. The (F2Y) symbol appears following those standards throughout the high school section.



K-8 Standards Coding



High School Standards Coding



Focus and Organization of the K-12 Domains and Conceptual Categories

The mathematics content of the *Wisconsin Standards for Mathematics* builds across grades and provides important underpinnings for the mathematics to be learned at subsequent levels. The coherence of the *Wisconsin Standards for Mathematics* lies in those connections, both within and across grade levels and topics. The graphic below illustrates the domains and conceptual categories of the *Wisconsin Standards for Mathematics*. The final row of the chart shows mathematical modeling as a process and how the teaching of that process can mature as students move through the grade bands.

<table>
<tr><th>GRADE</th><th>K</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>High School</th></tr>
<tr><td rowspan="6">DOMAINS/
CONCEPTUAL CATEGORIES</td><td>Counting and Cardinality</td><td colspan="8"></td><td rowspan="2">Algebra</td></tr>
<tr><td colspan="6">Operations and Algebraic Thinking</td><td colspan="3">Expressions and Equations</td></tr>
<tr><td colspan="6">Number and Operations in Base Ten</td><td colspan="3">The Number System</td><td>Number and Quantity</td></tr>
<tr><td colspan="3"></td><td colspan="3">Number and Operations Fractions</td><td colspan="2">Ratios and Proportional Relationships</td><td colspan="2">Functions</td></tr>
<tr><td colspan="6">Measurement and Data</td><td colspan="4">Statistics and Probability</td></tr>
<tr><td colspan="10">Geometry</td></tr>
<tr><td></td><td colspan="10"><<<<<<<<<<<<<<<<<<<<<<<<< Mathematical Modeling >>>>>>>>>>>>>>>>>>>>>>>>>></td></tr>
</table>

At the early elementary grades, the focus is largely on the areas of number and operations in base ten and algebraic thinking. This expands to a focus on fractions later in elementary school. The K-5 mathematics content provides the groundwork for the study of ratios, proportional reasoning, the number system, expressions and equations, and functions at the middle school level. By providing a focused mathematics experience in elementary and middle school, a strong foundation is developed for the content to be learned at the high school level. “Mathematical modeling should be taught at every stage of a student’s mathematical education” (Bliss and Libertini 2016, 7). “Mathematics is important for its own sake, but mostly...mathematics is important in dealing with the rest of the world. Certainly, mathematics will help students as they move on through school and into the world of work, but it can and should help them in their daily lives and as informed citizens” (Bliss and Libertini 2016, 7).

Additional Resources follow both the Standards for Mathematical Practice and the Content Standards for grades K-12:

- **Tables** (Appendix 1)—Provides problem situation types and examples as well as the properties of operations, equality, and inequality. Notice that tables 1, 2A, and 2B have been updated in *Wisconsin Standards for Mathematics (2021).*
- **Glossary** (Appendix 2)—Supports the understanding of all grade level standards.
- **Wisconsin’s Shifts in Mathematics** (Appendix 3)—Provides additional guidance about the five shifts, describing how instruction and instructional materials might reflect those shifts.
- **Mathematical Modeling** (Appendix 4) —Provides additional guidance about mathematical modeling, including connections between equitable teaching and mathematical modeling, a description of the modeling process, content connections, and a starting point for developing a K-12 progression of modeling.

Section III

Discipline: Mathematics

Standards

These revised state standards (2021), demonstrate the belief that every student has the ability to develop deep mathematical understanding as a confident and capable learner. To achieve this and develop strong mathematical identities in the process, all students need access to grade-level standards. It is important that users of this document keep the five Wisconsin Key Shifts at the forefront of their minds while interacting with both the Standards for Mathematical Practice and the Standards for Mathematical Content, K-12.

Wisconsin Standards for Mathematics also provides schools and districts with opportunities to make local decisions about curriculum, materials, and assessments. In order to provide guidance for these decisions, efforts have been made to ensure the standards promote educational equity. Examples include:

- Intentionally using the aspects of fluency—flexibility and efficiency—to prioritize student understanding and strategic competence as a way to build toward mathematical proficiency.
- Explicitly highlighting subitizing as a critical skill that supports understanding quantity and numerical relationships as foundational knowledge of number and operations.
- Using mathematical modeling throughout K-12 mathematical experiences that allow for exploration of authentic math problems that arise in everyday lives to support each student's identity as a problem solver.

Standards for Mathematical Practice

The Standards for Mathematical Practice are central to the teaching and learning of mathematics. These practices describe the behaviors and habits of mind that are exhibited by students who are mathematically proficient. Mathematical understanding is the intersection of these practices and mathematics content. It is critical that the Standards for Mathematical Practice are embedded in daily mathematics instruction.

The Standards for Mathematical Practice include:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others (Gutiérrez 2017, 17-18).
4. Model with mathematics.

5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The *Wisconsin Standards for Mathematics* (2021) includes K-12 versions of the Standards for Mathematical Practice as well as grade band versions of these same standards. The K-12 Standards for Mathematical Practice illustrate that these habits of mind are consistent at all levels of mathematics and develop over time. The grade band versions of the Standards for Mathematical Practice aim to provide examples of how these practices might come to life within grade level content, providing additional clarity. The writing team benefited from earlier efforts to make the practice standards grade-band specific. For example, the work of Massachusetts provided a starting point (Massachusetts Department of Elementary and Secondary Education 2017).

Standards for Mathematical Content

The Standards for Mathematical Content describe the sequence of important mathematics content that students learn. They are a combination of procedures and understandings. These content standards are organized around domains and clusters, which are specified by grade level, kindergarten through grade 8, and by conceptual category at high school. The domains at all levels are based on research-based learning progressions detailing what is known about students' mathematical knowledge, skill, and understanding. The progressions build from grade to grade and topic to topic, providing K-12 focus and coherence, but these standards do not dictate curriculum or teaching methods. For example, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. Other important cross-grade themes that should be noted and investigated are concepts such as the role of units and unitizing, the properties of operations across arithmetic and algebra, operations and the problems they solve, transformational geometry, reasoning and sense-making, and modeling of and with mathematics.

The introductions at each K-8 grade level specify two to four key areas that are identified as critical areas of instruction and discuss the use of mathematical modeling in supporting students in becoming flexible users of mathematics at that grade level. At the high school level, the introductions describe the focus for each conceptual category, as well as the connections to other categories and domains.

Learning mathematics with understanding is a focus of the *Wisconsin Standards for Mathematics* (2021). Many of the Standards for Mathematical Content begin with the verb "understand" and are crucial for mathematical proficiency. It is generally agreed that students understand a concept in mathematics if they can use mathematical reasoning with a variety of representations and connections to explain the concept to someone else or apply the concept to another situation. This is how ‘understand’ should be interpreted when implementing the *Wisconsin Standards for Mathematics* (2021).

While the Standards for Mathematical Practice should be addressed with all of the Standards for Mathematical Content, the content standards that begin with the verb “understand” are a natural intersection between the two.

Understanding Mathematics

Wisconsin Standards for Mathematics (2021) define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

Wisconsin Standards for Mathematics (2021) set grade-specific standards but do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations. It is also beyond the scope of these standards to define the full range of supports appropriate for bilingual learners, English language learners, and for students with special needs. At the same time, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. *Wisconsin Standards for Mathematics* (2021) should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities reading should allow for use of Braille, screen reader technology, or other assistive devices, while writing should include the use of a scribe, computer, or speech-to-text technology. In a similar vein, speaking and listening should be interpreted broadly to include sign language. No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom. However, these standards do provide clear signposts along the way to the goal of college and career readiness for all students.

Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

K-12 Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Math Practice 2: Reason abstractly and quantitatively.

K-12 Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Math Practice 3: Construct viable arguments and appreciate and critique the reasoning of others.

K-12 Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. While communicating their own mathematical ideas is important, students also learn to be open to others' mathematical ideas. They appreciate a different perspective or approach to a problem and learn how to respond to those ideas, respecting the reasoning of others (Gutiérrez 2017, 17-18). Together, students make sense of the mathematics by asking helpful questions that clarify or deepen everyone's understanding.

Math Practice 4: Model with mathematics.

K-12 Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Math Practice 5: Use appropriate tools strategically.

K-12 Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Math Practice 6: Attend to precision.

K-12 Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Math Practice 7: Look for and make use of structure.

K-12 Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Math Practice 8: Look for and express regularity in repeated reasoning.

K-12 Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

K-5 Elementary Standards

This section includes K-5 specific Standards for Mathematical Practice as well as grade level introductions, overviews, and Standards for Mathematical Content for each grade, kindergarten through fifth.

K-5 Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

K-5 Mathematically proficient elementary students explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, instead of hunting for “key words” in a word problem, students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets. If students are not at first making sense of a problem or seeing a way to begin, they ask questions about what is happening in the problem that will help them get started. As they work, they continually ask themselves, “Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, they might replace multi-digit numbers in a word problem with single-digit numbers to better appreciate the quantities in the problem and how they relate.

Mathematically proficient students consider different solution pathways, both their own and those of other students, in order to identify and analyze connections among approaches. They can explain connections among physical models, pictures, diagrams, equations, verbal descriptions, tables, and graphs. Once students have a solution, they often check their answers to problems using a different approach.

Math Practice 2: Reason abstractly and quantitatively.

K-5 Mathematically proficient elementary students make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using visual representations or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent.

Mathematically proficient students know and flexibly use different properties of operations, numerical relationships, numbers, and geometric objects. They can contextualize an abstract problem by placing it in a context that they can then use to make sense of the mathematical ideas. For example, if a student chooses to evaluate the expression 13 x 25 mentally, the student might think of a context to help produce a strategy—for example, by thinking “Thirteen groups of 25 is like having 13 quarters.” This prompts a strategy of thinking "I know that 10 quarters is $2.50 and 3 quarters is $0.75. $2.50 and $0.75 is $3.25." In this example the student uses a context to think through a strategy for solving the problem, using their knowledge of money and of decomposing one factor based on place value (13 = 10+3). The student then uses the context to identify the solution to the original problem.

Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context. Consider the problem: A teacher wants to bring 10 pumpkins to school to decorate the classroom. Some are big pumpkins and some are small pumpkins. How many of each size pumpkin might the teacher bring to school? When students create the number sentence 4+6=10, they have decontextualized the problem and expressed it with numbers and symbols. When they can explain that the number sentence means, "4 big pumpkins plus 6 small pumpkins equals 10 pumpkins," they demonstrate their ability to recontextualize the numbers and equation back to the word problem.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others.

K-5 Mathematically proficient elementary students construct verbal and written mathematical arguments that explain the reasoning underlying a strategy, solution, or conjecture. Arguments might use concrete referents such as objects, drawings, diagrams, and actions. Arguments might also rely on definitions, previously established results, properties, or structures. For example, a student might argue that 1/5 >1/9 on the basis that one of 5 equal parts of a whole is larger than one of 9 equal parts of that whole, because with more equal parts, the size of each part must be smaller. Another example is reasoning that two different shapes have equal area because it has already been demonstrated that they are each half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with four equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

Mathematically proficient students present their arguments in the form of representations, actions on those representations, explanations in words (oral or written), or a combination of these three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (Math Practice 8). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, primary grade students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. Intermediate grade students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals.

While communicating their own mathematical ideas is important, elementary students also learn to be open to others' mathematical ideas. They appreciate a different perspective or approach to a problem and learn how to respond to those ideas, respecting the reasoning of others (Gutiérrez 2017, 17-18). Together, students make sense of the mathematics, asking helpful questions that clarify or deepen everyone's understanding, and reconsider their own arguments in response to the collaboration.

Math Practice 4: Model with mathematics.

K-5 “In the course of a student’s mathematics education, the word ‘model’ is used in many ways. Several of these, such as manipulatives, demonstration, role modeling, and conceptual models of mathematics, are valuable tools for teaching and learning. However, they are different from the practice of mathematical modeling. Mathematical modeling, both in the workplace and in school, uses mathematics to answer big, messy, reality-based questions (Bliss and Libertini 2016, 7). “

Mathematically proficient elementary students formulate their own problems that emerge from natural circumstances as they mathematize the world around them. They can identify the mathematical elements of a situation and generate questions that can be addressed using mathematics (e.g., noticing and wondering). Students dig into the context and make assumptions as they decide “what matters.” Mathematically proficient elementary students understand that there are multiple solutions to a modeling problem so they are working to find a solution rather than the solution. Students make judgements about what matters and assess the quality of their solution (Bliss and Libertini 2016, 10). They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving their mathematical modeling approach if it has not served its purpose. As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (Math Practice 2).

In the elementary grades, students encounter mathematical modeling opportunities each and every day at school and at home. Students might consider how the classroom’s set of blocks should be shared throughout recess time. Students might then need to make assumptions about how many blocks each student should have as well as the length of time each student should have the blocks. Once a solution is determined, students could be asked to refine their model by posing the question, “What if one of our friends will not be at recess?” Children might also be presented with a bag of apples and simply asked “Is this enough for our class/family?” or consider the question, “Is the carpet in our classroom big enough for our bodies?”

Note: Although physical objects and visuals can be used to model a situation, using these tools absent a contextual situation is not an example of Math Practice 4. For example, solving a word problem using counters or a tape diagram would not be modeling with mathematics, instead this is modeling the mathematics. Math Practice 4 is about engaging in solving authentic real-world problems.

Math Practice 5: Use appropriate tools strategically.

K-5 Mathematically proficient elementary students strategically consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (e.g., manipulatives, pencil and paper, rulers), conceptual tools (e.g., properties of operations, algorithms), drawings or diagrams (e.g., number lines, tally marks, tape diagrams, arrays, tables, graphs), and available technologies (e.g., calculators, online apps).

Mathematically proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations. Students choose tools that are relevant and useful to the problem at hand. For example, when determining how to measure length, students may compare the benefits of using non-standard units of measure (e.g., their own hands, paperclips) versus standard units and tools (e.g., an inch or centimeter ruler). As another example, when presented with 1002-3 or 101-98, students subtract strategically, which may involve reasoning, counting, or decomposing rather than using a written algorithm.

Math Practice 6: Attend to precision.

K-5 Mathematically proficient elementary students use precise language to communicate orally and in written form. They come to appreciate, understand, and use mathematical vocabulary not in isolation, but in the context of doing mathematical thinking and problem solving. They may start by using everyday language to express their mathematical ideas and gradually select words with greater clarity and specificity. For example, they may initially use the word “triangle” to refer only to equilateral triangles resting on their bases, but come to understand and use a more precise definition of a triangle as a closed figure with three straight sides. As another example, they may initially explain a solution by saying, “it works" without explaining what “it" means but later clarify their explanation with specific details.

In using mathematical representations, students provide appropriate labels to precisely communicate the meaning of their representations (e.g., charts, graphs, and drawings). When making mathematical arguments about a solution, strategy, or conjecture (Math Practice 3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations.

Students use mathematical symbols correctly and can describe the meaning of the symbols they use. For example, they use the equal sign consistently and appropriately. They state the meaning of the symbols they choose in relation to the problem at hand.

Students use tools and strategies (e.g., measuring tools, estimation) effectively, to maintain a level of precision that is appropriate to the situation. They specify units of measure where needed.

Perseverance and attention to detail are mathematical habits of mind; mathematically proficient students check for reasonableness and accuracy by solving a problem a second way, analyzing errors, and learning from them.

Math Practice 7: Look for and make use of structure.

K-5 Mathematically proficient elementary students use structures such as place value, the properties of operations, and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (Math Practice 8). When students use an algorithm to solve 53-17 in order to fully understand how to decompose the tens and ones, they must understand that 53 can be seen as 4 tens and 13 ones, not just 5 tens and 3 ones.

When younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16 x 9, they might apply the structure of place value and the distributive property to find the product: 16 x 9 = (10 + 6) x 9 = (10 x 9) + (6 x 9). To determine the volume of a 3 x 4 x 5 rectangular prism, students might see the structure of the prism as five layers of 3 x 4 arrays of cubes.

Students in elementary grades look for and make use of structure when they view expressions as objects to observe and interpret. For example, students might observe that 120 – 41 must be one less than 120 – 40 because “if you subtract one more, the result will be one less” (Math Practice 8). Students can interpret the expression 5 x 3 + 6 x 3 as “five groups of three and six more groups of three” or notice there are a total of 11 groups of 3.

A word problem that involves distributing 29 marbles among 4 vases could lead (Math Practice 4) to an equation model (29 – 1) ÷ 4 = 7, where the expression on the left-hand side not only has the value 7 but also suggests, based on its structure, a process of discarding 1 marble and dividing the rest of the marbles equally into 4 groups of 7.

Math Practice 8: Look for and express regularity in repeated reasoning.

K-5 Mathematically proficient elementary students look for and identify regularities as they solve multiple related problems. Students make and test conjectures, reason about and express these regularities as generalizations about structures and relationships, and then use those generalizations to solve problems (Math Practice 7).

For example, younger students might notice that when tossing two-color counters to find combinations of a given number, over time students will notice a pattern (commutative property of addition). For example, when tossing six 2-sided counters, they may get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow.

In the elementary grades students can recognize and use patterns to help them become flexible with addition. For example, given the number string below, students may recognize they can take one away from the 5 and add it to the first number to make a multiple of ten. They also may notice a pattern related to the first digit increasing by 10, therefore the answer increases by 10.

9+5 19+5 29+5 39+5

When drawing and representing fractions, students might notice a consistent relationship between the numerator and denominator of fractions that equal one half (e.g., that the numerator is half the denominator and the denominator is two times the numerator). They can generalize from these repeated examples that all fractions equal to one half show this relationship.

As students practice articulating their observations both verbally and in writing, they learn to communicate with greater precision (Math Practice 6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (Math Practice 3).

Introduction: Kindergarten

In Kindergarten, instructional time should focus on two critical areas: representing and comparing whole numbers, initially with sets of objects, and describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. "To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade" (Student Achievement Partners 2014).

Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set, counting out a given number of objects, comparing sets or numerals, and modeling simple joining and separating situations with sets of objects, or eventually with equations such as 5 + 2 = 7 and 7 – 2 = 5. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including subitizing or quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the primary grades, this often occurs as students mathematize their world. Students have opportunities for modeling in their play, classroom, and other everyday experiences as they explore messy, complex and non-routine problems. Students in different primary grades may consider the same modeling problem in different ways. For example, if a child looks at two books and comments that it will take longer to read one than the other, this presents the opportunity to consider, "How long will it take to read a book?" Students in earlier grades may simply consider heavier or taller books as those that will take longer to read while students in later grades may think about the number of pages in each book or additional characteristics that would impact the solution (Godbold, Malkevitch, Teague, and van der Kooij 2016, 51).

Students in the primary grades might also consider intentionally open questions, such as “How many carrots should be in a lunch?” or “Do the students in our class like carrots?” using mathematics to answer the question (Godbold, Malkevitch, Teague, and van der Kooij 2016, 45).

Grade K Overview

Counting and Cardinality

- Know number names and the count sequence.
- Tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11-19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in each category.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Kindergarten Content Standards

Counting and Cardinality (K.CC)

Cluster Statement	Notation	Standard
A. Know number names and the count sequence.	M.K.CC.A.1	Count to 100 by ones and by tens.
	M.K.CC.A.2	Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
	M.K.CC.A.3	Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).
B. Tell the number of objects.	M.K.CC.B.4	Understand the relationship between numbers and quantities; connect counting to cardinality. a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object (one to one correspondence). b. Understand that the last number name said tells the number of objects counted (cardinality). The number of objects is the same regardless of their arrangement or the order in which they were counted (number conservation). c. Understand that each successive number name refers to a quantity that is one larger and the previous number is one smaller (hierarchical inclusion).
	M.K.CC.B.5	Quickly recognize and name the quantity of up to 5 objects briefly shown in structured or unstructured arrangements without counting (perceptual subitizing).

NOTE: This cluster is continued on next page.

Counting and Cardinality (K.CC) (cont'd)

Cluster Statement	Notation	Standard
B. Tell the number of objects. (cont'd)	M.K.CC.B.6 [WI.2010. K.CC.B.5]	Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.
C. Compare numbers.	M.K.CC.C.7 [WI.2010. K.CC.C.6]	Identify whether the number of objects (up to 10) in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.
	M.K.CC.C.8 [WI.2010. K.CC.C.7]	Compare two numbers between 1 and 10 presented as written numerals using student-generated ways to record the comparison.

Operations and Algebraic Thinking (K.OA)

Cluster Statement	Notation	Standard
A. Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.	M.K.OA.A.1	Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, or numbers. Drawings need not show details but should show the mathematics in the problem.
	M.K.OA.A.2	Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. *See Appendix, Table 1 for specific problem situations and category information.*
	M.K.OA.A.3	Compose and decompose quantities within 10 a. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition with drawings or numbers. b. Quickly name the quantity of objects briefly shown in structured arrangements anchored to 5 (e.g., fingers, ten frames, math rack/rekenrek) with totals up to 10 without counting by recognizing the arrangement or seeing the quantity in subgroups that are combined (conceptual subitizing).
	M.K.OA.A.4	For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or numbers.
	M.K.OA.A.5	Flexibly and efficiently add and subtract within 5 using mental images and composing or decomposing numbers up to 5.

Number and Operations in Base Ten (K.NBT)

Cluster Statement	Notation	Standard
A. Work with numbers 11-19 to gain foundations for place value.	M.K.NBT.A.1	Compose and decompose numbers from 11 to 19 into 10 ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or numbers; understand that these numbers are composed of 10 ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data (K.MD)

Cluster Statement	Notation	Standard
A. Describe and compare measurable attributes.	M.K.MD.A.1	Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
	M.K.MD.A.2	Directly compare two objects with a measurable attribute in common, to see which object has "more of" / "less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller or shorter.
B. Classify objects and count the number of objects in each category.	M.K.MD.B.3	Classify objects into given categories; count the numbers of objects in each category and sort the categories by count. Limit category counts to be less than or equal to 10.

Geometry (K.G)

Cluster Statement	Notation	Standard
A. Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).	M.K.G.A.1	Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to.
	M.K.G.A.2	Correctly name shapes regardless of their orientations or overall size.
	M.K.G.A.3	Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").
B. Analyze, compare, create, and compose shapes.	M.K.G.B.4	Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
	M.K.G.B.5	Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
	M.K.G.B.6	Compose simple shapes to form larger shapes. *For example, "Can you join these two triangles with full sides touching to make a rectangle?"*

Introduction: Grade 1

In Grade 1, instructional time should focus on four critical areas: developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; developing understanding of whole number relationships and place value, including grouping in tens and ones; developing understanding of linear measurement and measuring lengths as iterating length units; and reasoning about attributes of, and composing and decomposing geometric shapes. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. "To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade" (Student Achievement Partners 2014).

Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding 2 is the same as counting on 2). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.

Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.

Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and gain experiences that allow them to intuit the transitivity principle for indirect measurement. (Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term. See the glossary for the definition of transitivity principle for indirect measurement.)

Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part–whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the primary grades, this often occurs as students mathematize their world. Students have opportunities for modeling in their play, classroom, and other everyday experiences as they explore messy, complex, and non-routine problems. Students in different primary grades may consider the same modeling problem in different ways. For example, if a child looks at two books and comments that it will take longer to read one than the other, this presents the opportunity to consider, "How long will it take to read a book?" Students in earlier grades may simply consider heavier or taller books as those that will take longer to read while students in later grades may think about the number of pages in each book or additional characteristics that would impact the solution (Godbold, Malkevitch, Teague and van der Kooij, 51). Students in the primary grades might also consider intentionally open questions, such as "How many carrots should be in a lunch?" or "Do the students in our class like carrots?", using mathematics to answer the question (Godbold, Malkevitch, Teague and van der Kooij 2016, 45).

Grade 1 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

Number and Operations in Base Ten

- Extend the counting sequence.
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 1 Content Standards

Operations and Algebraic Thinking (1.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving addition and subtraction.	M.1.OA.A.1	Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. *See Appendix, Table 1 for specific problem situations and category information.*
	M.1.OA.A.2	Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.
B. Understand and apply properties of operations and the relationship between addition and subtraction.	M.1.OA.B.3	Apply properties of operations as strategies to add and subtract. *Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Informal use of the commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Informal use of the associative property of addition.)*
	M.1.OA.B.4	Understand subtraction as an unknown-addend problem. *For example, subtract 10 - 8 by finding the number that makes 10 when added to 8.*

NOTE: This domain is continued on next page.

Operations and Algebraic Thinking (1.OA) (cont'd)

Cluster Statement	Notation	Standard
C. Add and subtract within 20.	M.1.OA.C.5	Use counting and subitizing strategies to explain addition and subtraction. a. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2). b. Use conceptual subitizing in unstructured arrangements with totals up to 10 and structured arrangements anchored to 5 or 10 (e.g., ten frames, double ten frames, math rack/rekenrek) with totals up to 20 to relate the compositions and decompositions to addition and subtraction.
	M.1.OA.C.6	Use multiple strategies to add and subtract within 20. a. Flexibly and efficiently add and subtract within 10 using strategies that may include mental images and composing and decomposing up to 10. b. Add and subtract within 20 using objects, drawings, or equations. Use multiple strategies that may include counting on; making a ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 - 4 = 13 - 3 - 1 = 10 - 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 - 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).
D. Work with addition and subtraction equations.	M.1.OA.D.7	Understand the meaning of the equal sign as "has the same value or amount as" and determine if equations involving addition and subtraction are true or false. *For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 - 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.*

Number and Operations in Base Ten (1.NBT)

Cluster Statement	Notation	Standard
A. Extend the counting sequence.	M.1.NBT.A.1	Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.
B. Understand place value.	M.1.NBT.B.2	Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: a. 10 can be thought of as a bundle of ten ones—called a "ten." b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
	M.1.NBT.B.3	Compare two two-digit numbers based on meanings of the tens and ones digits and describe the result of the comparison using words and symbols (>, =, and <).

NOTE: This domain is continued on next page.

Number and Operations in Base Ten (1.NBT) (cont'd)

Cluster Statement	Notation	Standard
C. Use place value understanding and properties of operations to add and subtract.	M.1.NBT.C.4	Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
	M.1.NBT.C.5	Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.
	M.1.NBT.C.6	Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Measurement and Data (1.MD)

Cluster Statement	Notation	Standard
A. Measure lengths indirectly and by iterating length units.	M.1.MD.A.1	Order three objects by length; compare the lengths of two objects indirectly by using a third object.
	M.1.MD.A.2	Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. *Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.*
B. Tell and write time.	M.1.MD.B.3	Tell and write time in hours and half-hours using analog and digital clocks.
C. Represent and interpret data.	M.1.MD.C.4	Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry (1.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.1.G.A.1	Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
	M.1.G.A.2	Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. Student use of formal names such as "right rectangular prism" is not expected.
	M.1.G.A.3	Partition circles and rectangles into two and four equal shares, describe and count the shares using the words *halves* and *fourths*, and use the phrases *half of* and *fourth of the whole*. Describe the whole as being two of the shares, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

Introduction: Grade 2

In Grade 2, instructional time should focus on four critical areas: extending understanding of base-ten notation, building flexibility and efficiency with addition and subtraction, using standard units of measure, and describing and analyzing shapes. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. “To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade” (Student Achievement Partners 2014).

Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1,000) written in base-ten notation, recognizing that the digits in each place represent amounts of hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

Students use their understanding of addition to develop flexibility and efficiency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.

Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.

Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the primary grades, this often occurs as students mathematize their world. Students have opportunities for modeling in their play, classroom, and other everyday experiences as they explore messy, complex and non-routine problems. Students in different primary grades may consider the same modeling problem in different ways. For example, if a child looks at two books and comments that it will take longer to read one than the other, this presents the opportunity to consider, "How long will it take to read a book?" Students in earlier grades may simply consider heavier or taller books as those that will take longer to read while students in later grades may think about the number of pages in each book or additional characteristics that would impact the solution (Godbold, Malkevitch, Teague, and van der Kooij 2016, 51). Students in the primary grades might also consider intentionally open questions, such as "How many carrots should be in a lunch?" or "Do the students in our class like carrots?" using mathematics to answer the question (Godbold, Malkevitch, Teague and van der Kooij 2016, 45).

Grade 2 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 2 Content Standards

Operations and Algebraic Thinking (2.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving addition and subtraction.	M.2.OA.A.1	Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. *See Appendix, Table 1 for specific problem situations and category information.*
B. Add and subtract within 20.	M.2.OA.B.2	Flexibly and efficiently add and subtract within 20 using multiple mental strategies which may include counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 - 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).
C. Work with equal groups of objects to gain foundations for multiplication.	M.2.OA.C.3	Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by twos; write an equation to express an even number as a sum of two equal addends.
	M.2.OA.C.4	Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten (2.NBT)

Cluster Statement	Notation	Standard
A. Understand place value.	M.2.NBT.A.1	Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens—called a "hundred". b. The numbers 100,200,300,400,500,600,700,800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
	M.2.NBT.A.2	Count within 1,000; skip-count by fives, tens, and hundreds.
	M.2.NBT.A.3	Read and write numbers to 1,000 using base-ten numerals, number names, and expanded form.
	M.2.NBT.A.4	Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, and describe the result of the comparison using words and symbols (>, =, and <).

NOTE: This domain is continued on next page.

Number and Operations in Base Ten (2.NBT) (cont'd)

Cluster Statement	Notation	Standard
B. Use place value understanding and properties of operations to add and subtract.	M.2.NBT.B.5	Flexibly and efficiently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. In Grade 2, subtraction with decomposition is an exception and may include drawings or representations.
	M.2.NBT.B.6	Add up to four two-digit numbers using strategies based on place value and properties of operations.
	M.2.NBT.B.7	Add and subtract within 1,000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.
	M.2.NBT.B.8	Mentally add 10 or 100 to a given number 100 - 900, and mentally subtract 10 or 100 from a given number 100 - 900.
	M.2.NBT.B.9	Explain why addition and subtraction strategies work, using place value and the properties of operations. These explanations may be supported by drawings or objects.

Measurement and Data (2.MD)

Cluster Statement	Notation	Standard
A. Measure and estimate lengths in standard units.	M.2.MD.A.1	Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
	M.2.MD.A.2	Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
	M.2.MD.A.3	Estimate lengths using units of inches, feet, centimeters, and meters.
	M.2.MD.A.4	Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.
B. Relate addition and subtraction to length.	M.2.MD.B.5	Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as number lines) and equations with a symbol for the unknown number to represent the problem.
	M.2.MD.B.6	Represent whole numbers as lengths from 0 on a number line with equally spaced points corresponding to the numbers 0, 1, 2 ... and represent whole-number sums and differences within 100 on a number line.

NOTE: This domain is continued on next page.

Measurement and Data (2.MD) (cont'd)

Cluster Statement	Notation	Standard
C. Work with time and money.	M.2.MD.C.7	Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.
	M.2.MD.C.8	Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using $ and ¢ symbols appropriately. *Example: If you have 2 dimes and 3 pennies, how many cents do you have?*
D. Represent and interpret data.	M.2.MD.D.9	Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.
	M.2.MD.D.10	Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put together, take-apart, and compare problems using information presented in a bar graph. *See Appendix, Table 1 for specific problem situations.*

Geometry (2.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.2.G.A.1	Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes. Sizes are compared directly or visually, not compared by measuring.
	M.2.G.A.2	Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
	M.2.G.A.3	Partition circles and rectangles into two, three, or four equal shares, describe and count the shares using the words *halves*, *thirds*, and *fourths*, and use phrases *half of*, *a third of*, and *a fourth of the whole*. Describe the whole as composed of two halves, three thirds, and four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Introduction: Grade 3

In Grade 3, instructional time should focus on four critical areas: developing understanding of multiplication and division and strategies for multiplication and division within 100; developing understanding of fractions, especially unit fractions (fractions with numerator 1); developing understanding of the structure of rectangular arrays and of area; and describing and analyzing two-dimensional shapes. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. “To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade” (Student Achievement Partners 2014).

Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.

Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, 1/2 of the paint in a small bucket could be less paint than 1/3 of the paint in a larger bucket, but 1/3 of a ribbon is longer than 1/5 of the same ribbon because when the ribbon is divided into three equal parts, the parts are longer than when the ribbon is divided into five equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the intermediate grades, students are eager to mathematize their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex and non-routine problems. Intermediate students might consider plans for visiting a city (Galluzzo, Levy, Long, and Zbiek 2016, 26). Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as "What should you bring for lunch?" At the intermediate level, students can play an important role in generating and defining the big modeling questions they would like to address. Students will be able to consider aspects of the problem such as nutritional information, quantities, taste preferences, and variety. The general question could become "What is the 'best' lunch?" (Godbold, Malkevitch, Teague, and van der Kooij 2016, 46).

Grade 3 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving multiplication and division.
- Understand properties of multiplication and the relationship between multiplication and division.
- Multiply and divide within 100.
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Number and Operations in Base Ten

- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Develop understanding of fractions as numbers.

Measurement and Data

- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- Represent and interpret data.
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

Geometry

- Reason with shapes and their attributes.

Standards for Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 3 Content Standards

Operations and Algebraic Thinking (3.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving multiplication and division.	M.3.OA.A.1	Interpret products of whole numbers, e.g., interpret 5 x 7 as the total number of objects in 5 groups of 7 objects each. *For example, describe a context in which a total number of objects can be expressed as 5 x 7.*
	M.3.OA.A.2	Interpret whole-number quotients of whole numbers, e.g., interpret 56 ÷ 8 as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. *For example, describe a context in which a number of shares or a number of groups can be expressed as 56 ÷ 8.*
	M.3.OA.A.3	Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. *See Appendix, Tables 2A and 2B for specific problem situations.*

NOTE: This domain is continued on next page.

Operations and Algebraic Thinking (3.OA) (cont'd)

Cluster Statement	Notation	Standard
B. Understand properties of multiplication and the relationship between multiplication and division.	M.3.OA.B.4 [WI.2010. 3.OA.B.5]	Apply properties of operations as strategies to multiply and divide. Student use of the formal terms for these properties is not necessary. *Examples: If 6 x 4 = 24 is known, then 4 x 6 = 24 is also known. (commutative property of multiplication.) 3 x 5 x 2 can be found by 3 x 5 = 15, then 15 x 2 = 30, or by 5 x 2 = 10, then 3 x 10 = 30. (associative property of multiplication.) Knowing that 8 x 5 = 40 and 8 x 2 = 16, one can find 8 x 7 as 8 x (5 + 2) = (8 x 5) + (8 x 2) = 40 + 16 = 56. (distributive property.)*
	M.3.OA.B.5 [WI.2010. 3.OA.B.6]	Understand division as an unknown-factor problem. *For example, find 32 ÷ 8 by finding the number that makes 32 when multiplied by 8.*
C. Multiply and divide within 100.	M.3.OA.C.6 [WI.2010. 3.OA.B.7]	Use multiplicative thinking to multiply and divide within 100. a. Use the meanings of multiplication and division, the relationship between the operations (e.g., knowing that 8 x 5 = 40, one could reason that 40 ÷ 5 = 8), and properties of operations (e.g., the distributive property) to develop and understand strategies to multiply and divide within 100. b. Flexibly and efficiently use strategies, the relationship between the operations, and properties of operations to find products and quotients with multiples of 0, 1, 2, 5, & 10 within 100.

NOTE: This domain is continued on next page.

Operations and Algebraic Thinking (3.OA) (cont'd)

Cluster Statement	Notation	Standard
D. Solve problems involving the four operations, and identify and explain patterns in arithmetic.	M.3.OA.D.7 [WI.2010. 3.OA.B.8]	Solve two-step word problems, posed with whole numbers and having whole number answers, using the four operations. Represent these problems using one or two equations with a letter standing for the unknown quantity. If one equation is used, grouping symbols (i.e. parentheses) may be needed. Assess the reasonableness of answers using mental computation and estimation strategies.
	M.3.OA.D.8 [WI.2010. 3.OA.B.9]	Identify arithmetic patterns (including patterns in the addition table or multiplication table) and explain them using properties of operations. *For example, observe that four times a number is always even, and explain why four times a number can be decomposed into two equal addends.*

Number and Operations in Base Ten (3.NBT)

Cluster Statement	Notation	Standard
A. Use place value understanding and properties of operations to perform multi-digit arithmetic, using a variety of strategies.	M.3.NBT.A.1	Use place value understanding to generate estimates for problems in real-world situations, with whole numbers within 1,000, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates (e.g., Is my estimate too low or too high? What degree of precision do I need for this situation?).
	M.3.NBT.A.2	Flexibly and efficiently add and subtract within 1,000 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
	M.3.NBT.A.3	Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 x 80, 5 x 60) using strategies based on place value and properties of operations.

Number and Operations—Fractions (3.NF)

Grade 3 assessment expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8, but students should have instructional experiences with other sized fractions.

Cluster Statement	Notation	Standard
A. Develop understanding of fractions as numbers.	M.3.NF.A.1	Understand a unit fraction as the quantity formed when a whole is partitioned into equal parts and explain that a unit fraction is one of those parts (e.g., 1/4). Understand fractions are composed of unit fractions. *For example, 7/4 is the quantity formed by 7 parts of the size 1/4.*
	M.3.NF.A.2	Understand and represent a fraction as a number on the number line. a. Understand the whole on a number line is defined as the interval from 0 to 1 and the unit fraction is defined by partitioning the interval into equal parts (i.e., equal-sized lengths). b. Represent fractions on a number line by iterating lengths of the unit fraction from 0. Recognize that the resulting interval represents the size of the fraction and that its endpoint locates the fraction as a number on the number line. *For example, 5/3 indicates the length of a line segment from 0 by iterating the unit fraction 1/3 five times and its end point locates the fraction 5/3 on the number line.*
	M.3.NF.A.3	Explain equivalence of fractions and compare fractions by reasoning about their size. a. Understand two fractions as equivalent (equal) if they are the same size or name the same point on a number line. b. Recognize and generate simple equivalent fractions, e.g., *1/2 = 2/4*, 4/6 = 2/3) and explain why the fractions are equivalent by using a visual fraction model (e.g., tape diagram or number line). c. Express whole numbers as fractions (3 = 3/1), and recognize fractions that are equivalent to whole numbers (4/4 = 1). *d.* d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Justify the conclusions by using a visual fraction model (e.g., tape diagram or number line) and describe the result of the comparison using words and symbols (>, =, and <).

Measurement and Data (3.MD)

Cluster Statement	Notation	Standard
A. Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.	M.3.MD.A.1	Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line.
	M.3.MD.A.2	Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l), excluding compound units such as cm^3 and finding the geometric volume of a container. Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem. *See Appendix, Table 2B for problem situations. Do not include multiplicative comparison problems.*
B. Represent and interpret data.	M.3.MD.B.3	Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. *For example, draw a bar graph in which each square in the bar graph might represent five pets.*
	M.3.MD.B.4	Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units—whole numbers, halves, fourths.

NOTE: This domain is continued on next page.

Measurement and Data (3.MD) (cont'd)

Cluster Statement	Notation	Standard
C. Geometric measurement: understand concepts of area and relate area to multiplication and to addition.	M.3.MD.C.5	Recognize area as an attribute of plane figures and understand concepts of area measurement. A square with side length 1 unit, called "a unit square" is said to have "one square unit" of area, and can be used to measure area. A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units.
	M.3.MD.C.6	Measure areas by counting unit squares (square cm, square m, square in, square ft., and improvised units).
	M.3.MD.C.7	Relate area to the operations of multiplication and addition. a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. b. Multiply side lengths to find areas of rectangles with whole number side lengths in the context of solving real-world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning. d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.

NOTE: This domain is continued on next page.

Measurement and Data (3.MD) (cont'd)

Cluster Statement	Notation	Standard
D. Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.	M.3.MD.D.8	Solve real-world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Geometry (3.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.3.G.A.1	Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
	M.3.G.A.2	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into four parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*

Introduction: Grade 4

In Grade 4, instructional time should focus on three critical areas: developing understanding, flexibility, and efficiency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. “To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade” (Student Achievement Partners 2014).

Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, and area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop flexibility with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., 15/9 = 5/3), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.

Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the intermediate grades, students are eager to mathematize their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex and non-routine problems. Intermediate students might consider plans for visiting a city (Galluzzo, Levy, Long, Zbiek 2016, 26). Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as “What should you bring for lunch?” At the intermediate level, students can play an important role in generating and defining the big modeling questions they would like to address. Students will be able to consider aspects of the problem such as nutritional information, quantities, taste preferences, and variety. The general question could become “What is the ‘best’ lunch?” (Godbold, Malkevitch, Teague, and van der Kooij 2016, 46).

Grade 4 Overview

Operations and Algebraic Thinking

- Use the four operations with whole numbers to solve problems.
- Gain familiarity with factors and multiples.
- Generate and analyze patterns.
- Multiply and divide within 100.

Number and Operations in Base Ten

- Generalize place value understanding for multi-digit whole numbers.
- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions and compare decimal fractions.

Measurement and Data

- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- Represent and interpret data.
- Geometric measurement: understand concepts of angle and measure angles.

Geometry

- Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 4 Content Standards

Operations and Algebraic Thinking (4.OA)

Cluster Statement	Notation	Standard
A. Use the four operations with whole numbers to solve problems.	M.4.OA.A.1	Interpret a multiplication equation as a multiplicative comparison, e.g., interpret 35 = 5 x 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
	M.4.OA.A.2	Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison. *See Appendix, Tables 2A and 2B.*
	M.4.OA.A.3	Solve multi-step word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies.
B. Gain familiarity with factors and multiples.	M.4.OA.B.4	Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.

NOTE: This domain is continued on next page.

Operations and Algebraic Thinking (4.OA) (cont’d)

Cluster Statement	Notation	Standard
C. Generate and analyze patterns.	M.4.OA.C.5	Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*
D. Multiply and divide within 100.	M.4.OA.D.6	Flexibly and efficiently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that 8 x 5 = 40, one knows 40 ÷ 5 = 8) or properties of operations [e.g., knowing that 7 x 6 can be thought of as 7 groups of 6 so one could think 5 groups of 6 is 30 and 2 more groups of 6 is 12 and 30 + 12 = 42 (informal use of the distributive property)].

Number and Operations in Base Ten (4.NBT)

Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

Cluster Statement	Notation	Standard
A. Generalize place value understanding for multi-digit whole numbers.	M.4.NBT.A.1	Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. *For example, recognize that 700 ÷ 70 = 10 by applying concepts of place value and division.*
	M.4.NBT.A.2	Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place and describe the result of the comparison using words and symbols (>, =, and <).
	M.4.NBT.A.3	Use place value understanding to generate estimates for real-world problem situations, with multi-digit whole numbers, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates. (e.g., Is my estimate too low or too high? What degree of precision do I need for this situation?)
B. Use place value understanding and properties of operations to perform multi-digit arithmetic.	M.4.NBT.B.4	Flexibly and efficiently add and subtract multi-digit whole numbers using strategies or algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
	M.4.NBT.B.5	Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, or area models.
	M.4.NBT.B.6	Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, or area models.

Number and Operations—Fractions (4.NF)

Grade 4 assessment expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100 but students should have instructional experiences with other sized fractions.

Cluster Statement	Notation	Standard
A. Extend understanding of fraction equivalence.	M.4.NF.A.1	Understand fraction equivalence. a. Explain why a fraction is equivalent to another fraction by using visual fraction models (e.g., tape diagrams and number lines), with attention to how the number and the size of the parts differ even though the two fractions themselves are the same size. b. Understand and use a general principle to recognize and generate equivalent fractions that name the same amount.
	M.4.NF.A.2	Compare fractions with different numerators and different denominators while recognizing that comparisons are valid only when the fractions refer to the same whole. Justify the conclusions by using visual fraction models (e.g., tape diagrams and number lines) and by reasoning about the size of the fractions, using benchmark fractions (including whole numbers), or creating common denominators or numerators. Describe the result of the comparison using words and symbols (>, =, and <).

NOTE: This domain is continued on next page.

Number and Operations—Fractions (4.NF) (cont'd)

Cluster Statement	Notation	Standard
B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.	M.4.NF.B.3	Understand composing and decomposing fractions. a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. b. Decompose a fraction into a sum of unit fractions or multiples of that unit fraction in more than one way, recording each decomposition by an equation. Justify decompositions with explanations, visual fraction models, or equations. *For example: 3/8 = 1/8 + 1/8 + 1/8; 3/8 = 1/8 + 2/8; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8.* c. Add and subtract fractions, including mixed numbers, with like denominators (e.g., 3/8 + 2/8) and related denominators (e.g., 1/2 +1/4, 1/3 + 1/6) by using visual fraction models (e.g., tape diagrams and number lines), properties of operations, and the relationship between addition and subtraction. d. Solve word problems involving addition and subtraction of fractions with like and related denominators, including mixed numbers, by using visual fraction models and equations to represent the problem. Students are not required to rename fractions in lowest terms nor use least common denominators.

NOTE: This cluster is continued on next page.

Number and Operations—Fractions (4.NF) (cont’d)

Cluster Statement	Notation	Standard
B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. (cont’d)	M.4.NF.B.4	Apply and extend previous understandings of multiplication to multiply a whole number times a fraction. a. Understand a fraction as a group of unit fractions or as a multiple of a unit fraction. *For example, 5/4 can be represented visually as 5 groups of 1/4, as a sum of unit fractions 1/4 + 1/4 + 1/4 + 1/4 + 1/4, or as a multiple of a unit fraction 5 x 1/4.* b. Represent a whole number times a non-unit fraction (e.g., 3 x 2/5) using visual fraction models and understand this as combining equal groups of the non-unit fraction (3 groups of 2/5) and as a collection of unit fractions (6 groups of 1/5), recognizing this product as 6/5. c. Solve word problems involving multiplication of a whole number times a fraction by using visual fraction models and equations to represent the problem. Understand a reasonable answer range when multiplying with fractions. *For example, if each person at a party will eat 3/8 of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?*
C. Understand decimal notation for fractions and compare decimal fractions.	M.4.NF.C.5	Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100. *For example, express 3/10 as 30/100, and add 3/10 + 4/100 = 34/100.*
	M.4.NF.C.6	Use decimal notation for fractions with denominators 10 or 100, connect decimals to real-world contexts, and represent with visual models (e.g., number line or area model). *For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line.*
	M.4.NF.C.7	Compare decimals to hundredths by reasoning about their size and using benchmarks. Recognize that comparisons are valid only when the decimals refer to the same whole. Justify the conclusions, by using explanations or visual models (e.g., number line or area model) and describe the result of the comparison using words and symbols (>, =, and <).

Measurement and Data (4.MD)

Cluster Statement	Notation	Standard
A. Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.	M.4.MD.A.1	Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb., oz.; l, ml; hr., min., sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table. *For example, know that 1 ft. is 12 times as long as 1 in. Express the length of a 4 ft. snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36).*
	M.4.MD.A.2	Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as a number line that feature a measurement scale.
	M.4.MD.A.3	Apply the area and perimeter formulas for rectangles in real-world and mathematical problems. *For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.*
B. Represent and interpret data.	M.4.MD.B.4	Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Solve problems involving addition and subtraction of fractions by using information presented in line plots. *For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.*

NOTE: This domain is continued on next page.

Measurement and Data (4.MD) (cont'd)

Cluster Statement	Notation	Standard
C. Geometric measurement: understand concepts of angle and measure angles.	M.4.MD.C.5	Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint and understand concepts of angle measurement: a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle" and can be used to measure angles. b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
	M.4.MD.C.6	Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
	M.4.MD.C.7	Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry (4.G)

Cluster Statement	Notation	Standard
A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.	M.4.G.A.1	Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.
	M.4.G.A.2	Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category and identify right triangles.
	M.4.G.A.3	Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Introduction: Grade 5

In Grade 5, instructional time should focus on three critical areas: developing flexible strategies with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); extending division to two-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing flexibility with strategies to compute whole number and decimal operations; and developing understanding of volume. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. "To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade" (Student Achievement Partners 2014).

Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop flexible strategies in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)

Students develop an understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They compute efficiently with multi-digit addition, subtraction, multiplication, and division while flexibly applying strategies. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop flexible strategies in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. In the intermediate grades, students are eager to mathematize their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex, and non-routine problems. Intermediate students might consider plans for visiting a city (Galluzzo, Levy, Long, and Zbiek 2016, 26). Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as “What should you bring for lunch?” At the intermediate level, students can play an important role in generating and defining the big modeling questions they would like to address. Students will be able to consider aspects of the problem such as nutritional information, quantities, taste preferences, and variety. The general question could become “What is the ‘best’ lunch?” (Godbold, Malkevitch, Teague, and van der Kooij 2016, 46).

Grade 5 Overview

Operations and Algebraic Thinking

- Write and interpret numerical expressions.
- Analyze patterns and relationships.

Number and Operations in Base Ten

- Understand the place value system.
- Perform operations with multi-digit whole numbers and with decimals to hundredths.

Number and Operations—Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Measurement and Data

- Convert like measurement units within a given measurement system.
- Represent and interpret data.
- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

Geometry

- Graph points on the coordinate plane to solve real-world and mathematical problems.
- Classify two-dimensional figures into categories based on their properties.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 5 Content Standards

Operations and Algebraic Thinking (5.OA)

Cluster Statement	Notation	Standard
A. Write and interpret numerical expressions.	M.5.OA.A.1	Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
	M.5.OA.A.2	Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. *For example, express the calculation "add 8 and 7, then multiply by 2" as 2 x (8 + 7). Recognize that 3 x (18932 + 921) is three times as large as 18932 + 921, without having to calculate the indicated sum or product.*
B. Analyze patterns and relationships.	M.5.OA.B.3	Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

Number and Operations in Base Ten (5.NBT)

Cluster Statement	Notation	Standard
A. Understand the place value system.	M.5.NBT.A.1	Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
	M.5.NBT.A.2	Explain patterns in the number of zeros of the product when multiplying a number by powers of 10 and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
	M.5.NBT.A.3	Read, write, and compare decimals to thousandths. a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., 347.392 = 3 x 100 + 4 x 10 + 7 x 1 + 3 x (1/10) + 9 x (1/100) + 2 x (1/1000). b. Compare decimals to thousandths based on meanings of the digits in each place and describe the result of the comparison using words and symbols (>, =, and <).
	M.5.NBT.A.4	Use place value understanding to generate estimates for problems in real-world situations, with decimals, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates (e.g. Is my estimate too low or too high? What degree of precision do I need for this situation?)

NOTE: This domain is continued on next page.

Number and Operations in Base Ten (5.NBT) (cont'd)

Cluster Statement	Notation	Standard
B. Perform operations with multi-digit whole numbers and with decimals to hundredths.	M.5.NBT.B.5	Flexibly and efficiently multiply multi-digit whole numbers using strategies or algorithms based on place value, area models, and the properties of operations.
	M.5.NBT.B.6	Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, or area models.
	M.5.NBT.B.7	Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.

Number and Operations—Fractions (5.NF)

Students are not required to rename fractions in lowest terms nor use least common denominators.

Cluster Statement	Notation	Standard
A. Use equivalent fractions as a strategy to add and subtract fractions.	M.5.NF.A.1	Add and subtract fractions and mixed numbers using flexible and efficient strategies, including renaming fractions with equivalent fractions. Justify using visual models (e.g., tape diagrams or number lines) and equations. *For example, 2/3 + 5/4 = 8/12 + 15/12 = 23/12.*
	M.5.NF.A.2	Solve word problems involving addition and subtraction of fractions referring to the same whole using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. *For example, recognize an incorrect result 2/5 + 1/2 = 3/7, by observing that 3/7 <1/2.*
B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions.	M.5.NF.B.3	Interpret a fraction as an equal sharing division situation, where a quantity (the numerator) is divided into equal parts (the denominator). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, by using visual fraction models (e.g., tape diagrams or area models) or equations to represent the problem. *For example, when 3 wholes are shared equally among 4 people each person has a share of size 3/4. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?*

NOTE: This cluster is continued on next page.

Number and Operations—Fractions (5.NF) (cont'd)

Cluster Statement	Notation	Standard
B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions. (cont'd)	M.5.NF.B.4	Apply and extend previous understandings of multiplication to multiply a fraction times a whole number (e.g., 2/3 x 4) or a fraction times a fraction (e.g., 2/3 x 4/5), including mixed numbers. a. Represent word problems involving multiplication of fractions using visual models to develop flexible and efficient strategies. *For example, use a visual fraction model to show (2/3) × 4 = 8/3, and create a story context for this equation. Do the same with (2/3) × (4/5) = 8/15.* b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles and represent fraction products as rectangular areas.
	M.5.NF.B.5	Interpret multiplication as scaling (resizing) by estimating whether a product will be larger or smaller than a given factor on the basis of the size of the other factor, without performing the indicated multiplication. a. Explain why multiplying a given number by a fraction greater than 1 results in a product greater than the given number and explain why multiplying a given number by a fraction less than 1 results in a product smaller than the given number. b. Relate the principle of fraction equivalence to the effect of multiplying or dividing a fraction by 1 or an equivalent form of 1 (e.g., 3/3, 5/5).
	M.5.NF.B.6	Solve real-world problems involving multiplication of fractions and mixed numbers by using visual fraction models (e.g., tape diagrams, area models, or number lines) and equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.

NOTE: This cluster is continued on next page.

Number and Operations – Fractions (5.NF) (cont'd)

Cluster Statement	Notation	Standard
B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions. (cont'd)	M.5.NF.B.7	Apply and extend previous understandings of division to divide unit fractions by whole numbers (e.g., 1/3 ÷ 4) and whole numbers by unit fractions (e.g., 4 ÷ 1/5). Students able to multiply fractions can develop strategies to divide fractions by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade. a. Interpret and represent division of a unit fraction by a non-zero whole number as an equal sharing division situation. For example, create a story context for (1/3) ÷ 4, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that (1/3) ÷ 4 = 1/12 because (1/12) × 4 = 1/3. b. Interpret and represent division of a whole number by a unit fraction as a measurement division situation. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4. *c.* Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions by using visual fraction models and equations to represent the problem. *For example, how much chocolate will each person get if 4 people share 1/3 lb. of chocolate equally? Each person gets 1/12 lb. of chocolate. How many 1/5-cup servings are in 4 cups of raisins? There are 20 servings of size 1/5-cup of raisins.*

Measurement and Data (5.MD)

Cluster Statement	Notation	Standard
A. Convert like measurement units within a given measurement system.	M.5.MD.A.1	Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.
B. Represent and interpret data.	M.5.MD.B.2	Make a line plot to display a data set of measurements in fractions of a unit (1/2, 1/4, 1/8). Use operations on fractions for this grade to solve problems involving information presented in line plots. *For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.*

NOTE: This domain is continued on next page.

Measurement and Data (5.MD) (cont'd)

Cluster Statement	Notation	Standard
C. Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.	M.5.MD.C.3	Recognize volume as an attribute of solid figures and understand concepts of volume measurement. a. A cube with side length 1 unit, called a "unit cube," is said to have "one cubic unit" of volume, and can be used to measure volume. b. A solid figure which can be packed without gaps or overlaps using *n* unit cubes is said to have a volume of *n* cubic units.
	M.5.MD.C.4	Measure volumes by counting unit cubes, using cubic cm, cubic in., cubic ft., and improvised units.
	M.5.MD.C.5	Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume. a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication. b. Apply the formulas $V = l \times w \times h$ and $V = B \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole number edge lengths in the context of solving real-world and mathematical problems. c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real-world problems.

Geometry (5.G)

Cluster Statement	Notation	Standard
A. Graph points on the coordinate plane to solve real-world and mathematical problems.	M.5.G.A.1	Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate).
	M.5.G.A.2	Represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane and interpret coordinate values of points in the context of the situation.
B. Classify two-dimensional figures into categories based on their properties.	M.5.G.B.3	Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. *For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.*
	M.5.G.B.4	Classify two-dimensional figures in a hierarchy based on properties.

6-8 Middle School Standards

This section includes 6-8 specific Standards for Mathematical Practice as well as grade level introductions, overviews, and Standards for Mathematical Content for each grade, six through eight.

6-8 Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

6-8 Mathematically proficient middle school students set out to understand a problem and then look for entry points to its solution. Students identify questions to ask and make observations about the situation by using noticing and attending to aspects of the problem that look familiar. Students make assumptions where needed to make the problem more clearly defined. They analyze problem conditions and goals, translating, for example, verbal descriptions into equations, diagrams, or graphs as part of the process. They consider analogous problems, and try special cases and simpler forms of the original in order to gain insight into its solution. For example, to understand why a 20 percent discount followed by a 20 percent markup does not return an item to its original price, they might translate the situation into a tape diagram or a general equation; or they might first consider the situation for an item priced at $100.

Mathematically proficient students can explain how alternate representations of problem conditions relate to each other. For example, they can identify connections between the solution to a word problem that uses only arithmetic and a solution that uses variables and algebra; and they can navigate among verbal descriptions, tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change.

Mathematically proficient students check their approach, continually asking themselves "Does this approach make sense?" and "Can I solve the problem in a different way?" Students ask themselves these types of questions as a way to preserve through problem solving. While working on a problem, they monitor and evaluate their progress and change course if necessary. Students will reflect and revise their solution as needed. They can understand the approaches of others to solving complex problems and compare approaches.

Math Practice 2: Reason abstractly and quantitatively.

6-8 Mathematically proficient middle school students make sense of quantities and relationships in problem situations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. They represent problem situations using symbols and then manipulate those symbols in search of a solution (decontextualize). They can, for example, solve problems involving unit rates by representing the situations in equation form. Mathematically proficient students also pause as needed during problem solving to validate the meaning of the symbols involved. In the process, they can look back at the applicable units of measure to clarify or inform solution pathways (contextualize). Students can integrate quantitative information and concepts expressed in text and visual formats. Students can examine the constant and coefficient used in a linear function and express the meaning of those numbers related to a contextual situation. They can work with the function in different representations, such as a graph, keeping in mind the slope and vertical intercept have meaning related to the context. Quantitative reasoning also entails knowing and flexibly using different properties of operations and objects. For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others.

6-8 Mathematically proficient middle school students understand and use assumptions, definitions, and previously established results in constructing verbal and written arguments. They understand the importance of making and exploring the validity of conjectures. They can recognize and appreciate the use of counterexamples. For example, using numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$. Conversely, given a pair of equivalent algebraic expressions, they can show that the two expressions name the same number regardless of which value is substituted into them by showing which properties of operations can be applied to transform one expression into the other. They can explain and justify their conclusions to others using numerals, symbols, and visuals. They also reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example, they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies.

While communicating their own mathematical ideas is important, middle school students also learn to be open to others' mathematical ideas. They appreciate a different perspective or approach to a problem and learn how to respond to those ideas, respecting the reasoning of others (Gutiérrez 2017, 17-18). Together, students make sense of the mathematics, asking helpful questions such as "How did you get that?", "Why is that true?", and "Does that always work?" that clarify or deepen everyone's understanding. Mathematically proficient students are able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explain what it is. Students engage in collaborative discussions, drawing on evidence from problem texts and arguments of others, follow conventions for collegial discussions, and qualify their own views in light of evidence presented. They can present their findings and results to a given audience through a variety of formats such as posters, whiteboards, and interactive materials.

Math Practice 4: Model with mathematics.

6-8 "In the course of a student's mathematics education, the word 'model' is used in many ways. Several of these, such as manipulatives, demonstration, role modeling, and conceptual models of mathematics, are valuable tools for teaching and learning. However, they are different from the practice of mathematical modeling. Mathematical modeling, both in the workplace and in school, uses mathematics to answer big, messy, reality-based questions” (Bliss and Libertini 2016, 7).

Mathematically proficient middle school students formulate their own problems that emerge from natural circumstances as they mathematize the world around them. They can identify the mathematical elements of a situation and generate questions that can be addressed using mathematics (e.g., noticing and wondering). Middle school students can see a complicated problem and understand how that problem contains smaller problems to be solved. They are comfortable making assumptions as they decide “what matters.” Mathematically proficient middle students understand that there are multiple solutions to a modeling problem so they are working to find a solution rather than the solution. Students make judgements about what matters and assess the quality of their solution (Bliss and Libertini 2016, 10). They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving their mathematical modeling approach if it has not served its purpose. As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (Math Practice 2).

In the middle school grades, students encounter mathematical opportunities each and every day at school and at home. Mathematically proficient middle school students might consider how to plan a route to get to school with their friends. Students might then need to make assumptions about travel time and when to leave the house. In the morning they implement their plan and revise it by changing the departure time or including additional friends to the route. As a classroom, students might plan a fundraising event involving selling popcorn after school. In this example, sometimes students will be engaged in only a part of the modeling cycle such as making assumptions about how much to charge or how much popcorn to make (Godbold, Malkevitch, Teague, and van der Kooij 2016, 50).

Note: Although physical objects and drawings can be used to model a situation, using these tools absent a contextual situation is not an example of Math Practice 4. For example, drawing an area model to illustrate the distributive property in $4(t + s) = 4t + 4s$ would not be an example of Math Practice 4. Math Practice 4 is about applying math to a problem in context.

Math Practice 5: Use appropriate tools strategically.

6-8 Mathematically proficient middle school students strategically consider the available tools when solving a mathematical problem and while exploring a mathematical relationship. These tools might include pencil and paper, concrete models, a ruler, a protractor, a dynamic graphing tool, a spreadsheet, a statistical package, or dynamic geometry software. Proficient students make sound decisions about when each of these tools might be helpful, recognizing both the insights to be gained and their limitations. For example, they use estimation to check reasonableness; graph functions defined by expressions to picture the way one quantity depends on another; use algebra tiles to see how the properties of operations familiar from the elementary grades continue to apply to algebraic expressions; use an area model to visualize multiplication of rational numbers; use dynamic graphing tools to approximate solutions to systems of equations; use spreadsheets to analyze data sets of realistic size; or use dynamic geometry software to discover properties of parallelograms. Students are also strategic about when not to use tools, such as by simplifying an expression before substituting values into it (Math Practice 7), or rounding the inputs to a calculation and calculating on paper when an approximate answer is enough (Math Practice 6). When making mathematical models, students know that technology can enable them to visualize the results of their assumptions, to explore consequences, and to compare predictions with data (Math Practice 4). Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

Math Practice 6: Attend to precision.

6-8 Mathematically proficient middle school students communicate precisely to others both verbally and in writing. They present claims and findings, emphasizing salient points in a focused, coherent manner with relevant evidence, sound and valid reasoning, well-chosen details, and precise language. They use clear definitions in discussion with others and in their own reasoning and determine the meaning of symbols, terms, and phrases as used in specific mathematical contexts. For example, they can use the definition of a rational number to explain why $\sqrt{2}$ is irrational. Middle school students describe congruence and similarity in terms of transformations in the plane. They decide which parts of a problem need to be defined by a variable, state the meaning of the symbols, consistently and appropriately, such as independent and dependent variables of a linear equation. They are careful about specifying units of measure and label axes to display the correct correspondence between quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate to the context. For example, they accurately apply scientific notation to large numbers and use measures of center to describe data sets. In statistics and probability, students must attend to precision in the manner they write their statistical questions, the manner in which they collect their data, and the process they use to develop a simulation. Mathematically proficient middle school students care that an answer is right or reasonable; they attend to precision when they check their work; they solve the problem another way; they make revisions where appropriate.

Math Practice 7: Look for and make use of structure.

6-8 Mathematically proficient middle school students look closely to discern a pattern or structure. They might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation $3x = 2y$ represents a proportional relationship with a unit rate of $3/2 = 1.5$. They might recognize how the Pythagorean Theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism. They also can step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by six. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines or understanding 1.05a as an original value, a, plus 5% of that value, 0.05a. They can evaluate numeric expressions without combining each term in the order they are given. For example, when a student evaluates the expression 2 - (-) - 2 .

Math Practice 8: Look for and express regularity in repeated reasoning.

6-8 Mathematically proficient middle school students notice if calculations are repeated and look for both general methods and general and efficient methods. Working with tables of equivalent ratios, they might deduce the corresponding multiplicative relationships and make generalizations about the relationship to rates. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity with which interior angle sums increase with the number of sides in a polygon might lead them to the general formula for the interior angle sum of an n-gon. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their results throughout all stages of the process.

Introduction: Grade 6

In Grade 6, instructional time should focus on four critical areas: connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems; completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; writing, interpreting, and using expressions and equations; and developing understanding of statistical thinking. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. “To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade” (Student Achievement Partners 2014).

Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus, students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.

Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.

Students understand the use of variables in mathematical expressions. They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.

Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically. Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. The median measures center in the sense that it is roughly the middle value. The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed equally, and also in the sense that it is a balance point. Students recognize that a measure of variability (interquartile range or mean absolute deviation) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability.

Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, and symmetry, considering the context in which the data were collected. Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the volume of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in Grade 7 by drawing polygons in the coordinate plane.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. The cluster statements that best provide opportunities to implement modeling problem(s) or task(s) are identified with an (M) following the statement. In the middle grades, students are eager to act independently in mathematizing their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex, and non-routine problems. Middle school students might consider plans for ordering pizza for the class. Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as "How many pizzas are needed?" or "What toppings should be ordered on the pizza?" At the middle school level, students can play an important role in generating and defining the big modeling questions they would like to address as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (Galluzzo, Levy, Long, and Zbiek 2016, 34).

Grade 6 Overview

Ratios and Proportional Relationships

- Understand ratio concepts and use ratio reasoning to solve problems. (M)

The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Flexibly and efficiently compute with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers. (M)

Expressions and Equations

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables. (M)

Geometry

- Solve real-world and mathematical problems involving area, surface area, and volume. (M)

Statistics and Probability

- Develop understanding of statistical variability. (M)
- Summarize and describe distributions. (M)

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 6 Content Standards

Ratios and Proportional Relationships (6.RP)

<table>
<tr><th>Cluster Statement</th><th>Notation</th><th>Standard</th></tr>
<tr><td rowspan="3">A. Understand ratio concepts and use ratio reasoning to solve problems. (M)</td><td>M.6.RP.A.1</td><td>Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.
For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."</td></tr>
<tr><td>M.6.RP.A.2</td><td>Understand the concept of a unit rate a/b associated with a ratio a:b with b ≠ 0, and use rate language in the context of a ratio relationship.
For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is 3/4 cup of flour for each cup of sugar." "We paid $75 for 15 hamburgers, which is a rate of $5 per hamburger."
Expectations for unit rates in this grade are limited to non-complex fractions.</td></tr>
<tr><td>M.6.RP.A.3</td><td>Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number lines, or equations.
a. Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios.
b. Solve unit rate problems including those involving unit pricing and constant speed.
For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?
c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.</td></tr>
</table>

The Number System (6.NS)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of multiplication and division to divide fractions by fractions.	M.6.NS.A.1	Interpret, represent, and compute division of fractions by fractions and solve word problems by using visual fraction models (e.g., tape diagrams, area models, or number lines), equations, and the relationship between multiplication and division. *For example, create a story context for (2/3) ÷ (3/4) such as "How many 3/4 -cup servings are in 2/3 of a cup of yogurt" or "How wide is a rectangular strip of land with length 3/4 mile and area 2/3 square mile?" Explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3.*
B. Flexibly and efficiently compute with multi-digit numbers and find common factors and multiples.	M .6.NS.B.2	Flexibly and efficiently divide multi-digit whole numbers using strategies or algorithms based on place value, area models, and the properties of operations.
	M.6.NS.B.3	Flexibly and efficiently add, subtract, multiply, and divide multi-digit decimals using strategies or algorithms based on place value, visual models, the relationship between operations, and the properties of operations.
	M.6.NS.B.4	Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor. *For example, express 36 + 8 as 4 (9 + 2).*
C. Apply and extend previous understandings of numbers to the system of rational numbers. (M)	M.6.NS.C.5	Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

NOTE: This cluster is continued on next page.

The Number System (6.NS) (cont'd)

Cluster Statement	Notation	Standard
C. Apply and extend previous understandings of numbers to the system of rational numbers. (M) (cont'd)	M.6.NS.C.6	Understand a rational number as a point on the number line. Extend number lines and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates. a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., -(-3) = 3, and that 0 is its own opposite. b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. c. Find and position integers and other rational numbers on a horizontal or vertical number line; find and position pairs of integers and other rational numbers on a coordinate plane.

NOTE: This cluster is continued on next page.

The Number System (6.NS) (cont'd)

Cluster Statement	Notation	Standard
C. Apply and extend previous understandings of numbers to the system of rational numbers. (M) (cont'd)	M.6.NS.C.7	Understand ordering and absolute value of rational numbers. a. Interpret statements of inequality as statements about the relative position of two numbers on a number line. For example, interpret -3 > -7 as a statement that -3 is located to the right of -7 on a number line oriented from left to right. b. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write -3 ℃ > -7 ℃ to express the fact that -3℃ is warmer than -7 ℃. c. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of -30 dollars, write |-30| = 30 to describe the size of the debt in dollars. d. Distinguish comparisons of absolute value from statements about order. *For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.*
	M.6.NS.C.8	Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.

The Expressions and Equations (6.EE)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of arithmetic to algebraic expressions.	M.6.EE.A.1	Write and evaluate numerical expressions involving whole-number exponents.
	M.6.EE.A.2	Write, read, and evaluate expressions in which letters stand for numbers. *a.* Write expressions that record operations with numbers and with letters standing for numbers. *For example, express the calculation "Subtract y from 5" as 5 - y.* *b.* Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. *For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.* *c.* Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (order of operations). *For example, use the formulas* $V = s^3$ *and* $A = 6\,s^2$ *to find the volume and surface area of a cube with sides of length* $s = 1/2$.
	M.6.EE.A.3	Apply the properties of operations to generate equivalent expressions. *For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.*
	M.6.EE.A.4	Identify when two expressions are equivalent (e.g., when the two expressions name the same number regardless of which value is substituted into them). *For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for.*

NOTE: This domain is continued on next page.

The Expressions and Equations (6.EE) (cont'd)

Cluster Statement	Notation	Standard
B. Reason about and solve one-variable equations and inequalities.	M.6.EE.B.5	Understand solving an equation or inequality as a process of answering a question: Which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
	M.6.EE.B.6	Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number or, depending on the purpose at hand, any number in a specified set.
	M.6.EE.B.7	Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q, and x are all nonnegative rational numbers.
	M.6.EE.B.8	Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.
C. Represent and analyze quantitative relationships between dependent and independent variables. (M)	M.6.EE.C.9	Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. *For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.*

Geometry (6.G)

Cluster Statement	Notation	Standard
A. Solve real-world and mathematical problems involving area, surface area, and volume. (M)	M.6.G.A.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
	M.6.G.A.2	Find volumes of right rectangular prisms with fractional edge lengths by using physical or virtual unit cubes. Develop (construct) and apply the formulas $V = l\,w\,h$ and $V = B\,h$ to find volumes of right rectangular prisms in the context of solving real-world and mathematical problems.
	M.6.G.A.3	Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
	M.6.G.A.4	Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics and Probability (6.SP)

Cluster Statement	Notation	Standard
A. Develop understanding of statistical variability. (M)	M.6.SP.A.1	Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. *For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.*
	M.6.SP.A.2	Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.
	M.6.SP.A.3	Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.
B. Summarize and describe distributions. (M)	M.6.SP.B.4	Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
	M.6.SP.B.5	Summarize numerical data sets in relation to their context, such as by: a. Reporting the number of observations. b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement. c. Describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered and the quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation) were given. d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

Introduction Grade 7

In Grade 7, instructional time should focus on four critical areas: developing understanding of and applying proportional relationships; developing understanding of operations with rational numbers and working with expressions and linear equations; solving problems involving scale drawings and informal geometric constructions, and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and drawing inferences about populations based on samples. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. "To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade" (Student Achievement Partners 2014).

Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about scale drawings by relating corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.

Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division to all rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. By applying these properties, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. They use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems.

Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. The cluster statements that best provide opportunities to implement modeling problem(s) or task(s) are identified with an (M) following the statement. In the middle grades, students are eager to act independently in mathematizing their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex, and non-routine problems. Middle school students might consider plans for ordering pizza for the class. Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as “How many pizzas are needed?” or “What toppings should be ordered on the pizza?” At the middle school level, students can play an important role in generating and defining the big modeling questions they would like to address as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (Galluzzo, Levy, Long, and Zbiek 2016, 34).

Grade 7 Overview

Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems. (M)

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations. (M)

Geometry

- Draw, construct ,and describe geometrical figures and describe the relationships between them.
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. (M)

Statistics and Probability

- Use random sampling to draw inferences about a population. (M)
- Draw informal comparative inferences about two populations. (M)
- Investigate chance processes and develop, use, and evaluate probability models. (M)

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 7 Content Standards

Ratios and Proportional Relationships (7.RP)

Cluster Statement	Notation	Standard
A. Analyze proportional relationships and use them to solve real-world and mathematical problems. (M)	M.7.RP.A.1	Compute unit rates associated with ratios of fractions, including ratios of lengths, areas, and other quantities measured in like or different units. *For example, if a person walks 1/2 mile in each 1/4 hour, compute the unit rate as the complex fraction ½ / ¼ miles per hour, equivalently 2 miles per hour.*
	M.7.RP.A.2	Recognize and represent proportional relationships between quantities. a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. c. Represent proportional relationships by equations. *For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as t = pn.* d. Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate.
	M.7.RP.A.3	Use proportional relationships to solve multi-step ratio and percent problems. *Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.*

The Number System (7.NS)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.	M.7.NS.A.1	Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line. *a.* Describe situations in which opposite quantities combine to make 0. *For example, if you earn \$10 and then spend \$10, you are left with \$0.* b. Understand $p + q$ as the number located a distance $\|q\|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts. c. Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts. d. Apply properties of operations as strategies to add and subtract rational numbers.
	M.7.NS.A.2	Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (-1)(-1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts. b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts. c. Apply properties of operations as strategies to multiply and divide rational numbers. d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

NOTE: This cluster is continued on next page.

The Number System (7.NS) (cont'd)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers. (cont'd)	M.7.NS.A.3	Solve real-world and mathematical problems involving the four operations with rational numbers. (Note: Computations with rational numbers extend the rules for manipulating fractions to complex fractions.)

The Expressions and Equations (7.EE)

Cluster Statement	Notation	Standard
A. Use properties of operations to generate equivalent expressions.	M.7.EE.A.1	Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
	M.7.EE.A.2	Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. *For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."*
B. Solve real-life and mathematical problems using numerical and algebraic expressions and equations. (M)	M.7.EE.B.3	Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. *For example: If a woman making $25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or $2.50, for a new salary of $27.50.*
	M.7.EE.B.4	Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p, q, and r are specific rational numbers. Flexibly and efficiently apply the properties of operations and equality to solve equations of these forms. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. *For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?* b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem. *For example: As a salesperson, you are paid $50 per week plus $3 per sale. This week you want your pay to be at least $100. Write an inequality for the number of sales you need to make and describe the solutions.*

Geometry (7.G)

Cluster Statement	Notation	Standard
A. Draw, construct, and describe geometrical figures and describe the relationships between them.	M.7.G.A.1	Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
	M.7.G.A.2	Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
	M.7.G.A.3	Describe the two-dimensional figures that result from slicing three dimensional figures parallel to the base, as in plane sections of right rectangular prisms and right rectangular pyramids.
B. Solve real-life and mathematical problems involving angle measure, area, surface area, and volume. (M)	M.7.G.B.4	Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
	M.7.G.B.5	Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
	M.7.G.B.6	Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Statistics and Probability (7.SP)

Cluster Statement	Notation	Standard
A. Use random sampling to draw inferences about a population. (M)	M.7.SP.A.1	Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.
	M.7.SP.A.2	Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. *For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.*
B. Draw informal comparative inferences about two populations. (M)	M.7.SP.B.3	Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. *For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.*
	M.7.SP.B.4	Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. *For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.*

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Statistics and Probability (7.SP) (cont'd)

Cluster Statement	Notation	Standard
C. Investigate chance processes and develop, use, and evaluate probability models. (M)	M.7.SP.C.5	Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around 1/2 indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
	M.7.SP.C.6	Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. *For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.*
	M.7.SP.C.7	Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy. a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. *For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.* b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. *For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies*

NOTE: This cluster is continued on next page.

Statistics and Probability (7.SP) (cont'd)

Cluster Statement	Notation	Standard
C. Investigate chance processes and develop, use, and evaluate probability models. (M) (cont'd)	M.7.SP.C.8	Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs. b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event. c. Design and use a simulation to generate frequencies for compound events. *For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?*

Introduction: Grade 8

In Grade 8, instructional time should focus on three critical areas: formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation and solving linear equations and systems of linear equations; grasping the concept of a function and using functions to describe quantitative relationships; analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem. Not all content in a given grade is emphasized equally in the standards. Some concepts and skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. “To say that some things have greater emphasis is not to say that anything in the standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade” (Student Achievement Partners 2014).

Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x-coordinate changes by an amount A, the output or y-coordinate changes by the amount $m \cdot A$. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and y-intercept) in terms of the situation. Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.

Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

One way to empower students in becoming flexible users of mathematics is to provide authentic real-world opportunities for them to engage in mathematical modeling. The cluster statements that best provide opportunities to implement modeling problem(s) or task(s) are identified with an (M) following the statement. In the middle grades, students are eager to act independently in mathematizing their world. Opportunities for modeling with mathematics are found within their school day as well as in their everyday experiences when they explore messy, complex and non-routine problems. Middle school students might consider plans for ordering pizza for the class. Students would grapple with many decisions and ways of mathematizing the situation as they put together a plan. Students might also consider intentionally open questions, such as "How many pizzas are needed?" or "What toppings should be ordered on the pizza?" At the middle school level, students can play an important role in generating and defining the big modeling questions they would like to address as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (Galluzzo, Levy, Long, and Zbiek 2016, 34).

Grade 8 Overview

The Number System

- Know that there are numbers that are not rational and approximate them by rational numbers.

Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connections between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations. (M)

Functions

- Define, evaluate, and compare functions.
- Use functions to model relationships between quantities. (M)

Geometry

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand and apply the Pythagorean Theorem. (M)
- Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. (M)

Statistics and Probability

- Investigate patterns of association in bivariate data. (M)

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Grade 8 Content Standards

The Number System (8.NS)

Cluster Statement	Notation	Standard
A. Know that there are numbers that are not rational and approximate them by rational numbers.	M.8.NS.A.1	Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually and use patterns to rewrite a decimal expansion that repeats into a rational number.
	M.8.NS.A.2	Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line, and estimate the value of expressions (e.g., ^{2}). *For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5 and explain how to continue on to get better approximations.*

The Expressions and Equations (8.EE)

Cluster Statement	Notation	Standard
A. Work with radicals and integer exponents.	M.8.EE.A.1	Know and apply the properties of integer exponents to generate equivalent numerical expressions. *For example,* $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.
	M.8.EE.A.2	Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
	M.8.EE.A.3	Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities and to express how many times as much one is than the other. *For example, estimate the population of the United States as* 3×10^8 *and the population of the world as* 7×10^9 *and determine that the world population is more than 20 times larger.*
	M.8.EE.A.4	Use technology to interpret and perform operations with numbers expressed in scientific notation. Choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading).
B. Understand the connections between proportional relationships, lines, and linear equations.	M.8.EE.B.5	Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. *For example, compare a distance-time graph to a distance–time equation to determine which of two moving objects has greater speed.*
	M.8.EE.B.6	Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.

NOTE: This domain is continued on next page.

The Expressions and Equations (8.EE) (cont'd)

Cluster Statement	Notation	Standard
C. Analyze and solve linear equations and pairs of simultaneous linear equations. (M)	M.8.EE.C.7	Solve linear equations in one variable. a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into equivalent forms. b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
C. Analyze and solve linear equations and pairs of simultaneous linear equations. (M)	M.8.EE.C.8	Analyze and solve pairs of simultaneous linear equations. a. Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. b. Solve systems of two linear equations in two variables by graphing and analyzing tables. Solve simple cases represented in algebraic symbols by inspection. *For example, $3x + 2y = 5$ and $3x + 2y = 6$ have no solution because $3x + 2y$ cannot simultaneously be 5 and 6.* c. Solve real-world and mathematical problems leading to two linear equations in two variables. *For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.*

Functions (8.F)

Cluster Statement	Notation	Standard
A. Define, evaluate, and compare functions.	M.8.F.A.1	Understand that a function is a rule that assigns to each input exactly one output. The graph of a numerically valued function is the set of ordered pairs consisting of an input and the corresponding output. Function notation is not required in Grade 8.
	M.8.F.A.2	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*
	M.8.F.A.3	Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function* $A = s^2$ *giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4), and (3,9), which are not on a straight line.*
B. Use functions to model relationships between quantities. (M)	M.8.F.B.4	Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models and in terms of its graph or a table of values.
	M.8.F.B.5	Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear, continuous or discrete). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Geometry (8.G)

Cluster Statement	Notation	Standard
A. Understand congruence and similarity using physical models, transparencies, or geometry software.	M.8.G.A.1	Verify experimentally the properties of rotations, reflections, and translations: a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines.
	M.8.G.A.2	Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
	M.8.G.A.3	Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
	M.8.G.A.4	Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
	M.8.G.A.5	Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. *For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line and give an argument in terms of transversals why this is so.*

NOTE: This domain is continued on next page.

Geometry (8.G) (cont'd)

Cluster Statement	Notation	Standard
B. Understand and apply the Pythagorean Theorem. (M)	M.8.G.B.6	Justify the relationship between the lengths of the legs and the length of the hypotenuse of a right triangle and the converse of the Pythagorean Theorem.
	M.8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	M.8.G.B.8	Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
C. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres. (M)	M.8.G.C.9	Know the relationship among the formulas for the volumes of cones, cylinders, and spheres (given the same height and diameter) and use them to solve real-world and mathematical problems.

Statistics and Probability (8.SP)

Cluster Statement	Notation	Standard
A. Investigate patterns of association in bivariate data. (M)	M.8.SP.A.1	Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.
	M.8.SP.A.2	Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line. and informally assess the model fit by judging the closeness of the data points to the line.
	M.8.SP.A.3	Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. *For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.*
	M.8.SP.A.4	Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. *For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?*

High School Standards

This section includes high school specific Standards for Mathematical Practice as well as the Standards for Mathematical Content. These revised state standards (2021) and research recognize that every student needs to have some common outcomes in the first two years of high school, denoted as (F2Y), as well as a third year of further mathematics. Together this completes the three years of mathematics required for graduation in Wisconsin state statute. While three years is the minimum, it is common for districts to offer four years of mathematics coursework. The (F2Y) standards continue developing students as mathematical thinkers building on their mathematical identity and agency from their work in middle school mathematics. These first two years also prepare students for a variety of options in a third or fourth year of mathematics. The third or fourth year of mathematics can be designed using a combination of standards not marked as (F2Y). The (+) symbol indicates that the standard is typically one that appears in advanced mathematics courses and is not viewed as mathematics necessary for a three-year graduation requirement.

Wisconsin Standards for Mathematics also provides schools/districts with opportunities to make local decisions about curriculum, materials, and assessments. In order to provide guidance for these decisions, efforts have been made to ensure the standards promote educational equity. Examples include:

- Intentionally considering that the standards prepare students for credit-bearing coursework in mathematics at two- and four- year institutions, as well as for other post-secondary options.
- Identifying standards for a common two-year mathematical experience designed to allow for exploration of real-life problems through mathematical modeling as a way to maximize career and college opportunities.

The high school content standards are listed in conceptual categories:

- Modeling
- Number and Quantity
- Algebra
- Functions
- Geometry
- Statistics and Probability

Conceptual categories portray a coherent view of high school mathematics. A student's engagement with functions, for example, crosses a number of course boundaries, potentially up through and including calculus.

High School Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

HS Mathematically proficient high school students analyze givens, constraints, relationships, and goals. They make assumptions where needed to make the problem more clearly articulated. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. Students identify questions to ask and make observations about the situation through notice and wondering strategies. While following a solution plan, they continually ask themselves, "Does this make sense?" They monitor and evaluate their progress and revise their plan as needed. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. High school students might, depending on the context of the problem, transform algebraic expressions to provide them with different information about the situation. They might look at a scatter plot of data or make sense of a situation and decide which family of functions is appropriate to use to model the contextual situation. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph and interpret representations of data, and search for regularity or trends. Mathematically proficient students gain deeper insight into problems by using a different approach, understanding the approaches of others to solve complex problems, and identifying correspondences between different approaches. Mathematically proficient students are engaged in the problem-solving process, do not give up when stuck, and accept that it is acceptable to proceed forward when confronted with confusion and struggle.

Math Practice 2: Reason abstractly and quantitatively.

HS Mathematically proficient high school students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. For example, high school students might work with an exponential function given in symbolic form, but connect how the symbols represent the context of an exponential growth or decay situation. Students are able to manipulate and change the form of the function and reveal different information about the situation based on the numbers stated in the algebraic representation. A student could give a short survey to a sample of students and record the responses in a spreadsheet to compute summary statistics. The computed statistics are then used to gain insight into the larger student population about the question of interest. In addition, students can write explanatory text that conveys their mathematical analyses and thinking, using relevant and sufficient facts, concrete details, quotations, and coherent discussion of ideas. When working with statistics, students need to move from the context to abstract quantities and decide which statistic or representation to use to describe that situation. Students can evaluate multiple sources of information presented in diverse formats (and media) to address a question or solve a problem. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others.

HS Mathematically proficient high school students understand and use stated assumptions, definitions, and previously established results in constructing verbal and written arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples and specific textual evidence to form their arguments. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. While communicating their own mathematical ideas is important, high school students also learn to be open to others' mathematical ideas. They appreciate a different perspective or approach to a problem and learn how to respond to those ideas, respecting the reasoning of others (Gutiérrez 2017, 17-18). Together, students make sense of the mathematics, asking helpful questions that clarify or deepen everyone's understanding, and reconsider their own arguments in response to the collaboration. Mathematically proficient students are able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is and why. They can construct formal arguments relevant to specific contexts and tasks. High school students learn to determine domains to which an argument applies. They can present their findings and results to a given audience through a variety of formats such as posters, whiteboards, and interactive materials.

Math Practice 4: Model with mathematics.

HS Mathematically proficient high school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. Students that engage in modeling have choice when solving problems. By high school, a student might use geometry to solve a design problem or build a function to describe how one quantity depends on another. Mathematically proficient high school students apply what they know and are comfortable making assumptions and approximations to simplify a complicated situation, realizing that their model will need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as dynamic software, diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. Students will investigate how changes in parameters can result in changes to the model. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, which result in improving the model to better serve its purpose. Students can carry out all phases of the modeling cycle as outlined in the Modeling Conceptual Category. Mathematically proficient high school students also retain the widely applicable techniques they first learned in middle school, such as proportional relationships, rates, and percentages, and apply these techniques as needed to real-world tasks of a complexity appropriate to high school.

Note: Although physical objects and visuals can be used to model a situation, using these tools absent a contextual situation is not an example of Math Practice 4. For example, using algebra tiles or an area model to illustrate factoring a quadratic expression would not be an example of practice standard Math Practice 4. Math Practice 4 is about applying math to a problem in context.

Math Practice 5: Use appropriate tools strategically.

HS Mathematically proficient high school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for high school to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students can use slider bars in a dynamic calculator in order to "what-if" a situation and see how the graph of a function changes when the parameters of the equation are changed. They can use a spreadsheet to model change when cells are dynamically linked together and values are changed. Students can analyze graphs of functions and solutions generated using a dynamic graphing device; they also know how to sketch graphs of common functions, choosing this approach over technology when a sketch will suffice (Math Practice 6). They detect possible errors by strategically using estimation and other mathematical knowledge, for example anticipating the general appearance of a graph of a function by identifying the structure of its defining expression (Math Practice 7). They are able to use software or websites to quickly generate data displays that would otherwise be time-consuming to construct by hand (such as histograms, box plots, or simulation models for random sampling). Students use technological tools to explore and deepen their understanding of mathematical concepts and analyze realistic data sets. They use dynamic geometry software to explore geometric transformations of figures using rigid and non-rigid motion. When making mathematical models, students know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data (Math Practice 4). Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

Math Practice 6: Attend to precision.

HS Mathematically proficient high school students communicate precisely to others both verbally and in writing, adapting their communication to specific contexts, audiences, and purposes. They increasingly use precise language, not only as a mechanism for effective communication, but also as a tool for understanding and solving problems. Describing their ideas precisely helps students understand the ideas in new ways. They use clear definitions in discussions with others and in their own reasoning. They state the meaning of the symbols that they choose. They are careful about specifying units of measure, labeling axes, defining terms and variables, and calculating accurately and efficiently with a degree of precision appropriate for the problem context. They present logical claims and counterclaims fairly and thoroughly in a way that anticipates the audiences' knowledge, concerns, and possible biases. High school students draw specific evidence from informational sources to support analysis, reflection, and research. They critically evaluate the claims, evidence, and reasoning of others and attend to important distinctions with their own claims or inconsistencies in competing claims. Students evaluate the conjectures and claims, data, analysis, and conclusions in texts that include quantitative elements, comparing those with information found in other sources. Diligence and attention to detail are mathematical virtues: Mathematically proficient students care that an answer is right; they minimize errors by keeping a long calculation organized; they check their work; they solve the problem another way; they make revisions where appropriate.

Math Practice 7: Look for and make use of structure.

HS Mathematically proficient high school students look closely to discern a pattern or structure. In the expression x2 + 9x + 14, high school students can see the 14 as 2 x 7, and the 9 as 2 + 7. In an equation, high school students recognize that 12 = 3(x-1)2 does not require distribution in the expression on the right in order to carry out the process of solving for x. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. Students can look closely for patterns and structure that arise in data sets, both large and small. "They use structure to separate the 'signal' from the 'noise' in a set of data—the 'signal' being the structure, the 'noise' being the variability" (Franklin, Bargagliotti, Case, Kader, Scheaffer, and Spangler 2015, 12). Students studying statistical inference will see an inherent structure when developing an estimation interval for a mean or proportion. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see 5 – 3(x – y)2 as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. Students make use of structure for a purpose, for example by applying the conclusion 5 – 3(x – y)2 5 in the context of an applied optimization problem. Students will notice that the structure of a quadratic function written symbolically in a-b-c form (standard), vertex form, or factored form will reveal different information about the graph of the function.

Math Practice 8: Look for and express regularity in repeated reasoning.

HS Mathematically proficient high school students notice if calculations are repeated, and look both for general and efficient methods. Noticing the regularity in the way terms sum to zero when expanding (x − 1)(x + 1), (x − 1)(x2 + x + 1), and (x − 1)(x3 + x2 + x + 1) might lead students to the general formula for the sum of a geometric series. Statistically proficient students maintain oversight of the statistical problem-solving process. They look for a repeatable process that helps to define a given statistic. "Students recognize that probability provides the foundation for identifying patterns in long-run variability, thereby allowing students to quantify uncertainty" (Franklin, Bargagliotti, Case, Kader, Scheaffer, and Spangler 2015, 12). Students recognize that statistics may change value if a random sampling process were repeated. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details and continually evaluating the reasonableness of their intermediate results. When students repeatedly compute products of the form (ax + b) (ax + b) and notice the pattern equals (a2x2 + 2abx + b2), they are looking for and expressing regularity in repeated reasoning. Students change their perspective or view and use what they know. They turn or break down structure to something they know. They solve tasks by solving a sub-problem or smaller version. They might add a line or turn geometric structures so they identify something they have worked with before. By doing this, it helps students move forward and not be stuck. Students use repetition in reasoning as they work with various expressions involving exponents and develop an understanding of the various structures.

High School—Modeling

Mathematical modeling is a messy, open-ended process that is immersed in the real world. It incorporates choice and decision making on the part of the student into the process, and it links classroom mathematics and statistics to everyday life. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, formulate suggestions, and present them to an audience in order to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology may be valuable for varying assumptions, exploring consequences, and comparing predictions with data.

Mathematical modeling is something all students should have access to and be exposed to throughout the mathematics curriculum in all of the courses they take in their high school experience. Mathematical modeling should not be thought of as something that only comes at the end of a unit, but rather a vehicle which can be used to teach new and meaningful mathematics at the beginning and middle of a unit. By engaging students in this process, it should disprove the belief that there is only one answer to a problem and instead encourage them to keep revising their thought process to refine their model in answering questions that are meaningful to them and their community.

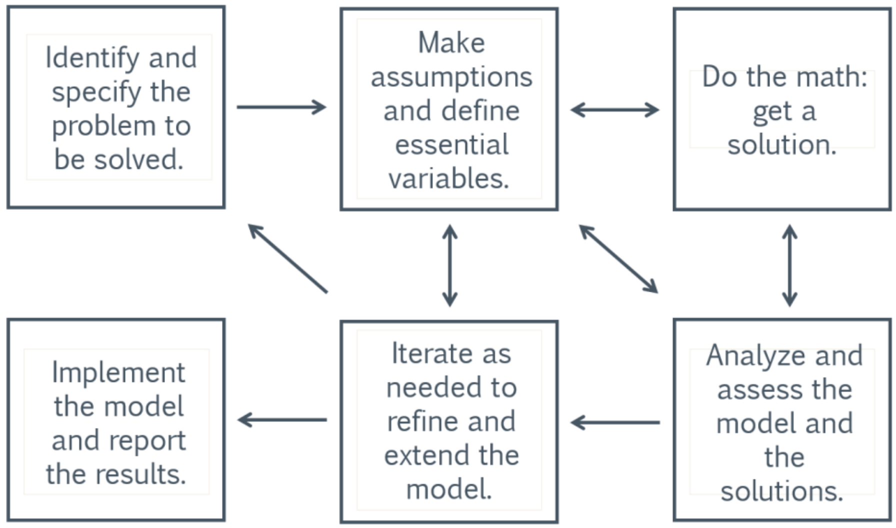
A model can be very simple, such as writing the total cost of producing an item as a product of unit price and number of items produced or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder or if a two-dimensional disk works well enough for our purposes. We have to invoke choice when deciding how to split the fare amongst three friends heading to different destinations when sharing the same taxi ride. Other situations—modeling a delivery route, developing a production schedule in a factory, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis. Formulating tractable models, representing such models, and analyzing and revising them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

Some examples of modeling situations might include:

- estimating a cell tower coverage when structural or electrical interference might play a significant factor;
- minimizing traffic congestion on a busy stretch of road and analyzing how far it will take to drive from one location to another;
- making the decision if it is better to focus on more three point shots than two point shots in basketball;
- designing the optimal location for a new school being built in a rural community;
- analyzing cost savings in driving across town to a different gas station for a cheaper price per gallon of fuel;
- deciding how to rank a variety of computers based on various attributes for a consumer tips report;
- designing a path to turnaround an aircraft at an airport; and
- comparing cost of ownership for two different types of vehicles.

In modeling situations, the models devised depend on a number of factors explored in the following questions. How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves; for example, when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



Source: GAIMME Report, 2016

The basic modeling cycle is summarized in the diagram (Bliss and Libertini 2016, 12-13). We can detail this diagram out starting with the upper left box and proceed clockwise:

1. We identify something in the real world we want to know, do, or understand. The result is a question in the real world.
2. We select "objects" that seem important in the real world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original question.
3. We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized question. This formulation is the model. We do the math to see what insights and results we get.

4. We consider: Does it address the problem? Does it make sense when translated back into the real world? Are the results practical, the answers reasonable, the consequences acceptable?
5. We iterate the process as needed to refine and extend our model.
6. For the real world, practical applications, we report our results to others and implement the solution (Bliss and Libertini 2016, 12-13).

Choices, assumptions, and approximations are present throughout the modeling cycle. It is not linear in nature, as the iterative process may have students return to a previous stage before going on to complete the modeling cycle. Functions, ratios and proportions, expressions and equations, descriptive and inferential statistical methods, probabilistic decision, and geometric representations are all important content tools for analyzing mathematical modeling problems. Dynamic graphing applications, spreadsheets, simulation applications, computer algebra systems, interactive applets, and dynamic geometry software are powerful tools that can be used within the mathematical modeling process.

Mathematical modeling is best completed and integrated into other content standards, and more especially in a group of standards within a cluster. Making mathematical models is the Fourth Standard for Mathematical Practice, and specific modeling standards appear throughout the high school content standards indicated by an (M) symbol. The (M) symbol appears next to various clusters of standards throughout all of the conceptual categories in high school mathematics. For instance, in the Functions Conceptual Category you will notice that within the domain of Linear, Exponential, and Quadratic Models there is a cluster of standards called "*Construct and compare linear, quadratic, and exponential models and solve problems,*" followed by (M). The denotation of (M) means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards.

Introduction: Number & Quantity

Numbers and Number Systems

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of number. At first, “number” means “counting number” (1, 2, 3...). Soon after that, 0 is used to represent “none” and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of numbers, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in three important ways: They have the commutative, associative, and distributive properties, and their new meanings are consistent with their previous meanings. Rational and real numbers are number systems utilized in the First Two Years (F2Y) while complex numbers are the new number system introduced in high school courses beyond the first two years.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5. Also, this example also is indicative of a continuous concept that may be introduced as part of a First Two Years (F2Y) and then continued as part of standards in later high school. For instance, students will use the cluster of standards, “*Extend the properties of exponents to rational exponents*,” from this conceptual category when interpreting and building exponential functions in the conceptual category of *Functions* to change the form of the exponential expression to gain further insight on a situation.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. Students have the opportunity to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents as part of the standards work beyond First Two Years (F2Y).

Quantities

In real-world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter First Two Years (F2Y) standards that address a wider variety of units in modeling, e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages.

They also encounter novel situations in which they themselves must conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process is sometimes called quantification. Quantification is important for science, as when surface area suddenly “stands out” as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Connection to Mathematical Modeling.
The (M) symbol appears next to various clusters throughout the Number and Quantity conceptual category. For example you will notice the (M) symbol within clusters of standards in the domains of Quantities and Vector and Matrix Quantities. This means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards when addressing them in the curriculum.

Number and Quantity Overview

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems. (M)

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.
- Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- Represent and model with vector quantities. (M)
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications. (M)

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The Real Number System (N-RN)

Cluster Statement	Notation	Standard
A. Extend the properties of exponents to rational exponents.	M.N.RN.A.1 (F2Y)	Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents.
	M.N.RN.A.2 (F2Y)	Rewrite expressions involving radicals and rational exponents using the properties of exponents.
B. Use properties of rational and irrational numbers.	M.N.RN.B.3	Explain why the sum or product of two rational numbers is rational, that the sum of a rational number and an irrational number is irrational, and that the product of a nonzero rational number and an irrational number is irrational.

Quantities (N-Q)

Cluster Statement	Notation	Standard
A. Reason quantitatively and use units to solve problems. (M)	M.N.Q.A.1 (F2Y)	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.
	M.N.Q.A.2 (F2Y)	Define appropriate quantities for the purpose of descriptive modeling.
	M.N.Q.A.3 (F2Y)	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities.

The Complex Number System (N-CN)

Cluster Statement	Notation	Standard
A. Perform arithmetic operations with complex numbers.	M.N.CN.A.1	Know there is a complex number *i* such that , and every complex number has the form *a* + *bi* with *a* and *b* real. Understand why complex numbers exist.
	M.N.CN.A.2	(+) Use the relation and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
	M.N.CN.A.3	(+) Find the conjugate of a complex number; use conjugates to find moduli (absolute values) and quotients of complex numbers.
B. Represent complex numbers and their operations on the complex plane.	M.N.CN.B.4	(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers) and explain why the rectangular and polar forms of a given complex number represent the same number.
	M.N.CN.B.5	(+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. *For example, because has modulus* 2 *and argument* 120°.
	M.N.CN.B.6	(+) Calculate the distance between numbers in the complex plane as the modulus of the difference and the midpoint of a segment as the average of the numbers at its endpoints.

NOTE: This domain is continued on next page.

The Complex Number System (N-CN) (cont'd)

Cluster Statement	Notation	Standard
C. Use complex numbers in polynomial identities and equations.	M.N.CN.C.7	Solve quadratic equations with real coefficients that have complex solutions. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.
	M.N.CN.C.8	(+) Extend polynomial identities to the complex numbers. *For example, rewrite* $x^2 + 4$ *as* $(x + 2i)(x - 2i)$.
	M.N.CN.C.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities (N-VM)

Cluster Statement	Notation	Standard
A. Represent and model with vector quantities. (M)	M.N.VM.A.1	(+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments and use appropriate symbols for vectors and their magnitudes (e.g., ***v***, |***v***|, ||***v***||, ***v***).
	M.N.VM.A.2	(+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
	M.N.VM.A.3	(+) Solve problems involving velocity and other quantities that can be represented by vectors.
B. Perform operations on vectors.	M.N.VM.B.4	(+) Add and subtract vectors. a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes. b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum. c. Understand vector subtraction v - w as v + (-w), where -w is the additive inverse of w, with the same magnitude as w and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order and perform vector subtraction component-wise.
	M.N.VM.B.5	(+) Multiply a vector by a scalar. a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as c(vx, vy) = (cvx, cvy). b. Compute the magnitude of a scalar multiple cv using ||cv|| = |c|v. Compute the direction of cv knowing that when |c|v 0, the direction of cv is either along v (for c > 0) or against v (for c < 0).

NOTE: This domain is continued on next page.

Vector and Matrix Quantities (N-VM) (cont'd)

Cluster Statement	Notation	Standard
C. Perform operations on matrices and use matrices in applications. (M)	M.N.VM.C.6	(+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
	M.N.VM.C.7	(+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
	M.N.VM.C.8	(+) Add, subtract, and multiply matrices of appropriate dimensions.
	M.N.VM.C.9	(+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation but still satisfies the associative and distributive properties.
	M.N.VM.C.10	(+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
	M.N.VM.C.11	(+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.
	M.N.VM.C.12	(+) Work with 2 x 2 matrices as transformations of the plane and interpret the absolute value of the determinant in terms of area.

Introduction: Algebra

Algebra is an area of high school that connects with both the theoretical and the applied side of mathematics. It strikes balance and complements many of the standards in the Functions conceptual category. It also extends to all of the other conceptual categories as algebraic thinking, syntax, and procedures are used within many of the cluster of standards in those areas. First Two Years (F2Y) standards include much of the work in the Seeing Structure in Expressions domain and the Creating Equations domain, with some work from the Reasoning with Equations and Inequalities domain. During these first two years, students predominantly work with linear, quadratic and exponential expressions, and equations. This foundational work connects nicely with the theoretical emphasis on work completed in the Arithmetic with Polynomials and Rational Expressions domain and many of the cluster of standards in the Reasoning with Equations and Inequalities domain that could come after the first two years.

The algebra studied in high school builds on the experiences students had with both expressions and equations throughout K-8 mathematics. Students need to understand the contextual meaning of an expression, while also being able to manipulate it into different forms to reveal various understanding about a situation. They also work with equations and inequalities where they understand the reasoning behind solving and how to rewrite equations in equivalent forms. The form of an algebraic expression and equation is important, depending on the purpose for using it. Different forms may be helpful when trying to better understand what a graphical representation will look like.

Expressions

An expression is a record of a computation with numbers, symbols that represent numbers and arithmetic operations. Students begin seeing and working with numerical and arithmetic expressions early in elementary mathematics. In Kindergarten through Grade 5, students use basic operations of addition, subtraction, multiplication, and division to build expressions. In Grades 6 through 8, the operations expand to using exponents with integers. Through the First Two Years (F2Y) standards, students are able to build upon the basic structure of expressions by adding sophistication through the use of expanded use of exponents and radicals. Within the First Two Years (F2Y) standards, students interpret expressions within context and examine equivalent forms of writing the expression, depending on the situation. As students work in this domain, their work is connected to the cluster of standards of Analyzing functions using different representations in the Function conceptual category. It is here that students will rewrite expressions in equivalent forms to reveal and explain different properties of functions such as quadratics and exponentials. Beyond the First Two Years (F2Y) standards, students write expressions in equivalent form using various techniques, like completing the square. They work with a geometric series to better understand situations such as computing mortgage monthly payments. A spreadsheet or a computer algebra system (CAS) can be used for these situations to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

By extending operations and number systems, students are able to complete more complex algebraic manipulation in attempting to make equivalent expressions that are more appropriate based upon the given context or situation. Algebraic manipulations are governed by the properties of operations that students encounter in Grades 6-8. These properties stem from The Number System and Expressions and Equations domains in Grades 6-8.

Equations and Inequalities

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form. Throughout the First Two Years (F2Y), students solve equations and inequalities involving one variable and create equations involving one or more variables. Techniques involving more complex manipulation are left for standards beyond First Two Years (F2Y). Many of these techniques are listed in the Arithmetic with Polynomials and Rational Expressions domain.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system. Students continue their work from 8th grade where they solve systems via inspection to using symbolic and graphical techniques within the First Two Years (F2Y) standards. Beyond those standards, students could use matrices to solve systems of equations.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers. Work with complex solutions is conducted within standards beyond First Two Years (F2Y).

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

When students create equations, they may contain two or more variables. Based on the situation, students decide how algebraic reasoning is used to solve for a variable of interest. These ideas are addressed in First Two Years (F2Y) standards and can be linked to standards that come later, such as work with inverse functions in the Building New Functions from Existing Functions cluster of standards from the Functions conceptual category.

The Algebra category is closely related to the Functions conceptual category. The concept of equivalent expressions can be understood in terms of functions. Two expressions are considered equivalent if they define the same function. An expression in one variable can be viewed as defining a function. Evaluating an expression for a given value is a similar process to finding a function's output for a given input. An equation in two variables can be seen as defining a function, if one of the variables is defined as input and the other output and if there is just one output for each input.

Connection to Mathematical Modeling

The (M) symbol appears next to various clusters throughout the Algebra conceptual category. For example, you will notice the (M) symbol within clusters of standards in the domains of Seeing Structure in Expressions and Creating Equations. This means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards when addressing them in the curriculum.

Algebra Overview

Seeing Structure in Expressions

- Interpret the structure of expressions. (M)
- Write expressions in equivalent forms to solve problems. (M)

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships. (M)

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Algebra Content Standards

Seeing Structure in Expressions (A-SSE)

Cluster Statement	Notation	Standard
A. Interpret the structure of expressions. (M)	M.A.SSE.A.1 (F2Y)	Interpret expressions that represent a quantity in terms of its context. a. Interpret parts of an expression, such as terms, factors, and coefficients. *For example, in the expression representing height of a projective, -16t2+vt+c recognizing there are three terms in the expression, factors within some of the terms, and coefficients. Interpret within the context the meaning of the coefficient -16 as related to gravity, the factor of v as the initial velocity, and the c-term as initial height.* b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret the expression representing population growth $P(1+r)^n$ as the product of P and a factor not depending on P. Interpret the meaning of the P-factor as initial population and the other factor as being related to growth rate and a period of time.*
	M.A.SSE.A.2 (F2Y)	Use the structure of an expression to identify ways to rewrite it. *For example, see x^4-y^4 as $(x^2)^2-(y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2-y^2)(x^2+y^2)$.*

NOTE: This domain is continued on next page.

Seeing Structure in Expressions (A-SSE) (cont'd)

Cluster Statement	Notation	Standard
B. Write expressions in equivalent forms to solve problems. (M)	M.A.SSE.B.3 (F2Y)	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example, if the expression 1.15^t represents growth in an investment account at time t (measured in years), it can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly rate of return is 1.2% based on an annual growth rate of 15%.*
	M.A.SSE.B.4	Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculating mortgage payments or tracking the amount of an antibiotic in the human body when prescribed for an infection.*

Arithmetic with Polynomials and Rational Expressions (A-APR)

Cluster Statement	Notation	Standard
A. Perform arithmetic operations on polynomials.	M.A.APR.A.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
B. Understand the relationship between zeros and factors of polynomials.	M.A.APR.B.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x – a$ is $p(a)$, so $p(a) = 0$ if and only if $(x – a)$ is a factor of $p(x)$.
	M.A.APR.B.3	Identify zeros of polynomials when suitable factorizations are available and use the zeros to construct a rough graph of the function defined by the polynomial.
C. Use polynomial identities to solve problems.	M.A.APR.C.4	Prove polynomial identities and use them to describe numerical relationships. *For example, use $(a+20)^2=a^2+40a+400$ to mentally or efficiently square numbers in the 20s. (e.g., $22^2=2^2+2*40+400=484$). Generalize to other double digit numbers. Use $a^2=(A+b)(a-b)+b^2$ and multiples of $a*10$ to square, e.g., $22^2=(22+12)(22-12)+12^2=340+144=484$. Recognize the visual representation of $(a+2b)^2-a^2=4ab$ as the area of a frame and find equivalent expressions.*
	M.A.APR.C.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.
D. Rewrite rational expressions.	M.A.APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system.
	M.A.APR.D.7	(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations (A-CED)

Cluster Statement	Notation	Standard
A. Create equations that describe numbers or relationships. (M)	M.A.CED.A.1 (F2Y)	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
	M.A.CED.A.2 (F2Y)	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
	M.A.CED.A.3 (F2Y)	Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*
	M.A.CED.A.4 (F2Y)	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange the formula $C = 5/9\ (F\text{-}32)$ so you solve for F.

Reasoning with Equations and Inequalities (A-REI)

Cluster Statement	Notation	Standard
A. Understand solving equations as a process of reasoning and explain the reasoning.	M.A.REI.A.1 (F2Y)	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	M.A.REI.A.2	Solve simple rational and radical equations in one variable and give examples showing how extraneous solutions may arise.
B. Solve equations and inequalities in one variable.	M.A.REI.B.3 (F2Y)	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
	M.A.REI.B.4 (F2Y)	Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, factoring, and graphing as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

NOTE: This domain is continued on next page.

Reasoning with Equations and Inequalities (A-REI) (cont’d)

Cluster Statement	Notation	Standard
C. Solve systems of equations.	M.A.REI.C.5 (F2Y)	Justify that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.
	M.A.REI.C.6 (F2Y)	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	M.A.REI.C.7 (F2Y)	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*
	M.A.REI.C.8	(+) Represent a system of linear equations as a single matrix equation in a vector variable.
	M.A.REI.C.9	(+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3 × 3 or greater).

NOTE: This domain is continued on next page.

Reasoning with Equations and Inequalities (A-REI) (cont'd)

Cluster Statement	Notation	Standard
D. Represent and solve equations and inequalities graphically.	M.A.REI.D.10 (F2Y)	Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
	M.A.REI.D.11 (F2Y)	Explain why the *x-coordinates* of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations.
	M.A.REI.D.12 (F2Y)	Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality) and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Introduction: Functions

Functions are an area of high school that connects heavily with the applied side of mathematics. It strikes balance and complements many of the standards in the Algebra conceptual category. It also extends to Geometry and Statistics & Probability as the notation and concepts are used within the cluster of standards in those areas. A function describes situations where an input value determines another and extends beyond being just a relationship. Functional relationships should not be thought of only in a numeric sense, as linking a person's social media account to their password would be an example. This example shows the vertical line test should not be the only tool used to decide if a relationship is a function.

Students should also work with functions through multiple representations (symbolic, tabular, graphical, spoken, or written words). As they use functions to model meaningful situations, they will move between and within representations to have a better understanding of the phenomena they are trying to understand. In a symbolic representation an algebraic expression like $f(x) = a + bx$; or a recursive rule might be used to relate a restaurant bill to the amount of money left for a tip of X%. Through a tabular representation, a student might see the relationship between weight and height of an individual. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's characteristics. A verbal representation might say, "I'll give you a date from last year, you give me the high temperature in a given city."

Throughout the First Two Years (F2Y) Functions standards, students build on their middle school work involving Ratios and Proportions (6th and 7th grade) and then in 8th grade are introduced to the function concept, though function notation is not required. During 8th grade, they will define, evaluate, and compare functions. They use functions to model relationships between quantities. The construction of these models are focused on linear, but they have experience describing qualitative features and characteristics of both linear and non-linear graphs.

During high school, students will interpret and build functions, which are both domains of study that link together throughout the high school experience. The interpretation of functions is something all students focus on within the clusters of First Two Years (F2Y) standards. They begin to use function notation and better understand the concepts and characteristics associated with them, such as domain and range. Students will explore various families of functions and use them to describe situations that occur in terms of a context. It is here where they will analyze them and move between and within different representations. The work within this domain complements the work within the building functions domain. Within these clusters of standards, students will model a relationship between two quantities and build new functions from existing functions. For instance, a new sub sandwich shop that is opening might want to model the average cost of producing each sandwich by using a cost function and dividing it by a function that represents how many sandwiches are produced. In another situation, a function that models temperature in a house as regulated by a programmable thermostat might be modified if there is a change in desired temperature or time for the heat source to turn on. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

The other area of focus in the Function category is the study of Families of Functions. This links together the Linear, Quadratic, and Exponential domain, with the Trigonometric Functions domain. There is also overlap with other functions that are described in the Algebra category. During the First Two Years (F2Y) cluster of standards, students will focus mainly on Linear, Quadratic and Exponential functions. They will interpret these functions and use them to model real-world phenomena. Appropriate dynamic technology should be used to assist students to see comparisons between the characteristics these families have and the distinctions that occur within parameters and form used to represent in symbolic form, patterns that are noticeable between successive outputs in a table, the structure of their graphs, and appropriate contextual situations that lend themselves to one family of function over another. As with the interpret and build domains, these families of functions standards provide avenues for students to engage in modeling.

It is later in their studies students would experience work with data that is periodic and repeats, such as the height or a rider on a Ferris wheel or the voltage cycle in an alternating current. Both of these could be modeled by a trigonometric function. It is here that students should continue to examine structure and see how the change in one parameter in the symbolic representation would change different characteristics of the function, such as midline or amplitude. Students would explore how a change in one representation would change the other representations. These ideas link back to the clusters of standards that students are exploring within the Building Functions domain. Once again, dynamic technology is appropriate for students to make connections between and within these representations. This entire conceptual category should be one of many areas of inspiration for integrating mathematical modeling problems and tasks during the high school mathematics experience.

Connection to Mathematical Modeling

The (M) symbol appears next to various clusters throughout the Function conceptual category. For example you will notice the (M) symbol within clusters of standards in the domains of Interpreting Functions, Building Functions, Linear, Exponential and Quadratic Models, and Trigonometric Functions. This means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards when addressing them in the curriculum.

Function Overview

Interpreting Functions

- Understand the concept of a function and use function notation.
- Intercept functions that arise in applications in terms of the context. (M)
- Analyze functions using different representations. (M)

Building Functions

- Build a function that models a relationship between two quantities. (M)
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems. (M)
- Interpret expressions for functions in terms of the situation they model. (M)

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions. (M)
- Prove and apply trigonometric identities.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions (F-IF)

Cluster Statement	Notation	Standard
A. Understand the concept of a function and use function notation.	M.F.IF.A.1 (F2Y)	Understand that a function from one set, discrete or continuous, (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range.
	M.F.IF.A.2 (F2Y)	Use function notation, evaluate functions. and interpret statements that use function notation in terms of a context.
	M.F.IF.A.3 (F2Y)	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, in an arithmetic sequence, f(x) = f(x-1) + C or in a geometric sequence, f(x) = f(x-1) * C, where C is a constant.*
B. Interpret functions that arise in applications in terms of context. (M)	M.F.IF.B.4 (F2Y)	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*
	M.F.IF.B.5	Relate the domain of a function to its graph and find an appropriate domain (discrete or continuous) in the context of the given problem.
	M.F.IF.B.6	Calculate and interpret the average rate of change of a linear or nonlinear function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

NOTE: This domain is continued on next page.

Interpreting Functions (F-IF) (cont'd)

Cluster Statement	Notation	Standard
C. Analyze functions using different representations. (M)	M.F.IF.C.7 M.F.IF.C.7a (F2Y)	Graph functions expressed symbolically and show key features of the graph using an efficient method. a. Graph linear and quadratic functions and show intercepts, maxima, and minima, and exponential functions, showing intercepts and end behavior. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. c. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior. d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available and showing end behavior. e. Graph logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	M.F.IF.C.8 (F2Y)	Write a function defined by an expression in equivalent forms to reveal and explain different properties of the function. a. Use an efficient process to rewrite $f(x) = ax^2+bx+c$ as $f(x) = a(x-h)^2 +k$ or $f(x)=a(x-p)(x-q)$ to determine the characteristics of the function and interpret these in terms of a context. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions, where t is in years, such as $y = (1.01)^{12t}$ is approximately $y = (1.127)^t$, where t is in years, meaning it is a 1% growth rate each month and a 12.7% growth rate each year. Identify percent rate of change in functions, where t is in years, such as $y = (1.2)^{(t/10)}$ is approximately $y = (1.018)^t$, meaning it is a 20% growth rate each decade and a 1.8% growth rate each year.*
	M.F.IF.C.9 (F2Y)	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions (F-BF)

Cluster Statement	Notation	Standard
A. Build a function that models a relationship between two quantities. (M)	M.F.BF.A1	Write a function that describes a relationship between two quantities. a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example: The temperature of a cup of coffee can be modeled by combining together a function representing difference in temperature and the actual room temperature, which results in an exponential model. An average cost function can be created by dividing the cost of purchasing n items by the number of n items purchased, which results in a rational function.* c. Work with composition of functions using tables, graphs, and symbols. *For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.*
	M.F.BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

NOTE: This domain is continued on next page.

Building Functions (F-BF) (cont'd)

Cluster Statement	Notation	Standard
B. Build new functions from existing functions.	M.F.BF.B.3	Identify the effect on the graph of replacing *f(x)* by *f(x) + k*, *k f(x)*, *f(kx)*, and *f(x + k)* using transformations for specific values of *k* (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
	M.F.BF.B.4	Identify and create inverse functions, using tables, graphs, and symbolic methods to solve for the other variable. *For example: Each car in a state is assigned a unique license plate number, and each license plate number is assigned to a unique car; thus there is an inverse relationship. Rearrange the formula C=59(F-32) so you solve for F. You examine a table of values and realize the inputs and outputs are invertible. Two graphs are symmetrical about the line y = x.*
	M.F.BF.B5	Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Linear, Quadratic, and Exponential Models (F-LE)

Cluster Statement	Notation	Standard
A. Construct and compare linear, quadratic, and exponential models and solve problems. (M)	M.F.LE.A.1 (F2Y)	Distinguish between situations that can be modeled with linear functions and with exponential functions. a. Prove that linear functions grow by equal differences over equal intervals and that exponential functions grow by equal factors over equal intervals. b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.
	M.F.LE.A.2 (F2Y)	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).
	M.F.LE.A.3 (F2Y)	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.
	M.F.LE.A.4	For exponential models, express as a logarithm the solution to $abc^{ct} = d$ where a, c, and d are numbers and the base b is 2,10, or e; evaluate the logarithm using technology.
B. Interpret expressions for functions in terms of the situation they model.	M.F.LE.B.5 (F2Y)	Interpret the parameters in a linear or exponential function in terms of a context.

Trigonometric Functions (F-TF)

Cluster Statement	Notation	Standard
A. Extend the domain of the trigonometric functions of the unit circle.	M.F.TF.A.1	Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
	M.F.TF.A.2	Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
	M.F.TF.A.3	(+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.
	M.F.TF.A.4	(+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
B. Model periodic phenomena with trigonometric functions. (M)	M.F.TF.B.5	Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.
	M.F.TF.B.6	(+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
	M.F.TF.B.7	(+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.
C. Prove and apply trigonometric identities.	M.F.TF.C.8	Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
	M.F.TF.C.9	(+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Introduction: Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Within the First Two Years (F2Y) standards, students focus on the domains of congruence, circles, similarity, and right triangle trigonometry, and measurement and dimension. As students move from the First Two Years (F2Y), modeling with geometry and geometric properties are addressed at both levels of the standards. Beyond the First Two Year students might work with conics and applications of trigonometry to general triangles. Later in college, some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Connection to Mathematical Modeling

The (M) symbol appears next to various clusters of standards throughout the Geometry conceptual category. For example, you will notice the (M) symbol within clusters in the domains of Similarity, Right Triangles, and Trigonometry, and Geometric Measurement and Dimension. This means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards when addressing them in the curriculum.

Geometry Overview

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Prove geometric theorems.
- Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles. (M)
- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems. (M)
- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations. (M)

Geometry

Congruence (G-CO)

Cluster Statement	Notation	Standard
A. Experiment with transformations in the plane.	M.G.CO.A.1 (F2Y)	Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
	M.G.CO.A.2 (F2Y)	Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
	M.G.CO.A.3 (F2Y)	Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
	M.G.CO.A.4 (F2Y)	Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
	M.G.CO.A.5 (F2Y)	Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

NOTE: This domain is continued on next page.

Congruence (G-CO) (cont'd)

Cluster Statement	Notation	Standard
B. Understand congruence in terms of rigid motion.	M.G.CO.B.6 (F2Y)	Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
	M.G.CO.B.7 (F2Y)	Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
	M.G.CO.B.8 (F2Y)	Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.
C. Prove geometric theorems.	M.G.CO.C.9 (F2Y)	Prove theorems about lines and angles. Theorems should include: Vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent, and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.
	M.G.CO.C.10 (F2Y)	Prove theorems about triangles. Theorems should include: Measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.
	M.G.CO.C.11 (F2Y)	Prove theorems about parallelograms. Theorems should include: Opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely; rectangles are parallelograms with congruent diagonals.

NOTE: This domain is continued on next page.

Congruence (G-CO) (cont'd)

Cluster Statement	Notation	Standard
D. Make geometric constructions.	M.G.CO.D.12 (F2Y)	Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.
	M.G.CO.D.13 (F2Y)	Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

Similarity, Right Triangles, and Trigonometry (G-SRT)

Cluster Statement	Notation	Standard
A. Understand similarity in terms of similarity trans-formations.	M.G.SRT.A.1 (F2Y)	Verify experimentally the properties of dilations given by a center and a scale factor: a. A dilation takes a line not passing through the center of the dilation to a parallel line and leaves a line passing through the center unchanged. b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
	M.G.SRT.A.2 (F2Y)	Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
	M.G.SRT.A.3 (F2Y)	Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.
B. Prove theorems involving similarity.	M.G.SRT.B.4 (F2Y)	Prove theorems about triangles. Theorems include: A line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.
	M.G.SRT.B.5 (F2Y)	Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

NOTE: This domain is continued on next page.

Similarity, Right Triangles, and Trigonometry (G-SRT) (cont'd)

Cluster Statement	Notation	Standard
C. Define trigonometric ratios and solve problems involving right triangles. (M)	M.G.SRT.C.6 (F2Y)	Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.
	M.G.SRT.C.7 (F2Y)	Explain and use the relationship between the sine and cosine of complementary angles.
	M.G.SRT.C.8 (F2Y)	Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.
D. Apply trigonometry to general triangles.	M.G.SRT.D.9	(+) Derive the formula $A = 1/2\ ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
	M.G.SRT.D.10	(+) Prove the Laws of Sines and Cosines and use them to solve problems.
	M.G.SRT.D.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles (G-C)

Cluster Statement	Notation	Standard
A. Understand and apply theorems about circles.	M.G.C.A.1 (F2Y) [WI.2010. G.C.A.2 and G.C.A.3]	Identify and describe relationships among inscribed angles, radii, and chords. Prove properties of angles for a quadrilateral inscribed in a circle. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.
B. Find arc lengths and areas of sectors of circles.	M.G.C.B.2 (F2Y) [WI.2010. G.C.B.5]	Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties (G-GPE)

Cluster Statement	Notation	Standard
A. Translate between the geometric description and the equation for a conic section.	M.G.GPE.A.1	Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
	M.G.GPE.A.2	(+) Derive the equation of a parabola given a focus and directrix.
	M.G.GPE.A.3	(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
B. Use coordinates to prove simple geometric theorems algebraically.	M.G.GPE.B.4 (F2Y)	Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, √3) lies on the circle centered at the origin and containing the point (0, 2).*
	M.G.GPE.B.5 (F2Y)	Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
	M.G.GPE.B.6 (F2Y)	Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
	M.G.GPE.B.7 (F2Y)	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).

Geometric Measurement and Dimension (G-GMD)

Cluster Statement	Notation	Standard
A. Explain volume formulas and use them to solve problems. (M)	M.G.GMD.A.1 (F2Y)	Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.
	M.G.GMD.A.2 (F2Y) [WI.2010. G.GMC.A.3]	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.
B. Visualize relationships between two-dimensional and three-dimensional objects.	M.G.GMD.B.3 (F2Y) [WI.2010. G.GMD.4]	Identify three-dimensional objects generated by rotations of two-dimensional objects.

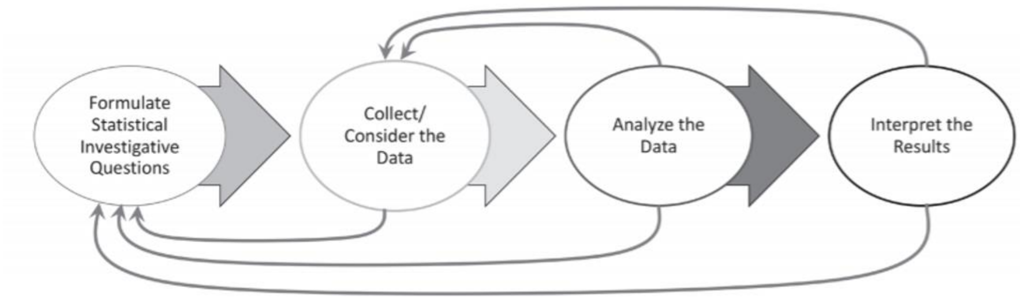
NOTE: This domain is continued on next page.

Geometric Measurement and Dimension (G-GMD) (cont'd)

Cluster Statement	Notation	Standard
C. Apply geometric concepts in modeling situations. (M)	M.G.GMD.C.4 (F2Y) [WI.2010. M.G.MG.A.1]	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
	M.G.GMD.C.5 (F2Y) [WI.2010. M.G.MG.A.2]	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).
	M.G.GMD.C.6 (F2Y) [WI.2010. M.G.MG.A.3]	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

Introduction: Statistics and Probability

First Two Years (F2Y) Statistics and Probability standards should be completed by all high school students. By meeting these standards, students are prepared for further course work addressing standards in the domains of Making Inferences and Justifying Conclusions and Use Probability to Make Decisions. This can lead students to further course work in statistics and data science. Students should be actively engaged in statistics through an investigative process that involves questioning, data collection, analysis, and interpretation



Source: Bargagliotti and Franklin 2020, 13

Decisions or predictions are often based on data—numbers in context. All statistical study should be situated within a context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Throughout the First Two Years (F2Y) Statistics standards, students build on the work they were introduced to in 6th and 8th grades, working with descriptive statistics and linear modeling. Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots (dot plots, box plots, and histograms). Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken. Students extend the idea of spread from using mean absolute deviation in middle school to interpreting the meaning of standard deviation and what it means for a sample set of data, as well as later in the context of a normal distribution.

Within the Interpret Linear Models cluster of First Two Years (F2Y) standards, students decide if a scatter plot is best modeled by a linear relationship using a correlation coefficient and interpret the meaning of the parameters in the linear model. They examine residuals when comparing observed values to predicted values. Students will also create quadratic and exponential models from scatter plots based on shape and context, which connects to work they did in Functions under the domain of Linear, Exponential, and Quadratic Models.

Throughout high school, students further develop their understanding of inferential statistics using simulation, which is grounded in their work from 7th grade. Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data. In critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Throughout the First Two Years (F2Y) probability standards, students build on the work they were introduced to in 7th grade as they work with random processes that can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. During 8th grade, students focused on looking at associations between two categorical variables displayed in a two-way table. Throughout the First Two Years (F2Y) standards their interpretation of these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of different representations such as two-way tables, Venn diagrams, and tree diagrams. Also, students use the probability tools they developed during 7th grade and in the First Two Years (F2Y) standards to make decisions about payoffs for a game, analyzing the testing of a medical product, or conduct DNA testing in criminal cases. This is addressed in the "*Use probability* to evaluate outcomes of decisions" cluster of standards.

Technology plays an important role in statistics and probability by making it possible to generate plots (dot plots, box plots, and histograms), regression functions, and correlation coefficients. Statistical software and animations allow students to simulate many possible outcomes in a short amount of time. Students might use bootstrapping techniques or randomization to meet standards within the make inferences and justify conclusions from sample surveys, experiments, and observational studies cluster of standards. Simulation can be used to build conceptual understanding of many of the ideas within inferential statistics, which can lead to a solid foundation for further study in future courses.

Connection to Mathematical Modeling

The (M) symbol appears next to various clusters of standards throughout the Statistics and Probability conceptual category. For example, you will notice the (M) symbol within clusters in the domains of Interpreting Categorical and Quantitative Data, Making Inferences and Justifying Conclusions, Conditional Probability and the Rules of Probability, and Using Probability to Make Decisions. This means that it would be appropriate to implement a modeling problem(s) or task(s) that relates to this group of standards when addressing them in the curriculum.

Statistics and Probability Overview

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable. (M)
- Summarize, represent, and interpret data on two categorical and quantitative variables. (M)
- Interpret linear models. (M)

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments. (M)
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies. (M)

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data. (M)
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems. (M)
- Use probability to evaluate outcomes of decisions. (M)

Standards for Mathematical Practice

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Statistics and Probability (SP)
Interpreting Categorical and Quantitative Data (S-ID)

Cluster Statement	Notation	Standard
A. Summarize, represent, and interpret data on a single count or measurement variable. (M)	M.SP.ID.A.1 (F2Y)	Represent data with plots on the real number line (dot plots, histograms, and box plots).
	M.SP.ID.A.2 (F2Y)	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
	M.SP.ID.A.3 (F2Y)	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
	M.SP.ID.A.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use statistical packages calculators, spreadsheets, and tables to estimate areas under the normal curve.

NOTE: This domain is continued on next page.

Interpreting Categorical and Quantitative Data (S-ID) (cont'd)

Cluster Statement	Notation	Standard
B. Summarize, represent, and interpret data on two categorical and quantitative variables. (M)	M.SP.ID.B.5 (F2Y)	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies as examples of proportionality and disproportionality). Recognize possible associations and trends in the data.
	M.SP.ID.B.6 (F2Y)	Represent data on two quantitative variables on a scatter plot and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize appropriate families of functions to model.* b. Informally assess the fit of a function by plotting and analyzing residuals. c. Fit a linear function for a scatter plot that suggests a linear association.
C. Interpret linear models (M)	M.SP.ID.C.7 (F2Y)	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
	M.SP.ID.C.8 (F2Y)	Use technology to create a correlation coefficient for a linear fit and then interpret its meaning for the model.
	M.SP.ID.C.9 (F2Y)	Distinguish between correlation and causation. *For example, cities with a higher number of fast food restaurants tend to have more hospitals. While there is a clear correlation (likely driven by population), we cannot conclude that fast food restaurants are causing more hospitals to open. There is a relationship between height and reading level in elementary school children. This is not because changes in height cause better reading ability. Rather as children get older, they get both taller and improve in their reading skills.*

Making Inferences and Justifying Conclusions (S-IC)

Cluster Statement	Notation	Standard
A. Understand and evaluate random processes underlying statistical experiments. (M)	M.SP.IC.A.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
	M.SP.IC.A.2	Decide if a specified model is consistent with results from a given data-generating process (e.g., using simulation). For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?
B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies. (M)	M.SP.IC.B.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
	M.SP.IC.B.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
	M.SP.IC.B.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
	M.SP.IC.B.6	Evaluate reports based on data.

Conditional Probability and the Rules of Probability (S-CP)

Cluster Statement	Notation	Standard
A. Understand independence and conditional probability and use them to interpret data. (M)	M.SP.CP.A.1 (F2Y)	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").
	M.SP.CP.A.2 (F2Y)	Understand that two events A and B are independent if the probability of *A* and *B* occurring together is the product of their probabilities and use this characterization to determine if they are independent.
	M.SP.CP.A.3 (F2Y)	Understand the conditional probability of *A* given *B* as *P(A and B)/P(B)* and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.
	M.SP.CP.A.4 (F2Y)	Represent data from two categorical variables using two-way frequency tables or Venn diagrams. Interpret the representation when two categories are associated with each object being classified. Use the representation as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.*
	M.SP.CP.A.5 (F2Y)	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.*

NOTE: This domain is continued on next page.

Conditional Probability and the Rules of Probability (S-CP) (cont'd)

Cluster Statement	Notation	Standard
B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.	M.SP.CP.B.6 (F2Y)	Use a representation such as a two-way table or Venn diagram to find the conditional probability of A given B as the fraction of B's outcomes that also belong to A and interpret the answer in terms of the model.
	M.SP.CP.B.7 (F2Y)	Use a representation such as a two-way table or Venn diagram to apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.
	M.SP.CP.B.8	(+) Use a representation such as a tree diagram to apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B\|A) = P(B)P(A\|B)$, and interpret the answer in terms of the model.
	M.SP.CP.B.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Using Probability to Make Decisions (S-MD)

Cluster Statement	Notation	Standard
A. Calculate expected values and use them to solve problems. (M)	M.SP.MD.A.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.
	M.SP.MD.A.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.
	M.SP.MD.A.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. *For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices and find the expected grade under various grading schemes.*
	M.SP.MD.A.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. *For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?*

NOTE: This domain is continued on next page.

Using Probability to Make Decisions (S-MD) (cont'd)

Cluster Statement	Notation	Standard
B. Use probability to evaluate outcomes of decisions. (M)	M.SP.MD.B.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. a. Find the expected payoff for a game of chance. *For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.* b. Evaluate and compare strategies on the basis of expected values. *For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.*
	M.SP.MD.B.6	Use probabilities to make fair decisions (e.g., drawing for a party door prize where attendees earn one entry to the drawing for each activity they complete, using an electronic spinner to pick a team spokesperson at random from a group, flip a coin to decide which of two friends gets to choose the movie, using a random number generator to select people to include in a sample for an experiment).
	M.SP.MD.B.7	Analyze decisions and strategies using probability concepts (e.g., balancing expected gains and risk, medical product testing, choosing an investment option, deciding when to kick an extra point vs. two point conversion after a touchdown in football).

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Appendix I

Tables

“There’s a moment in problem solving...when you...classify a problem and make connections between this problem and the catalog of problems you’ve tackled in the past” (Su 2020, 36).

Table 1. Addition and Subtraction Situations by Grade Level

(Adapted from *Progressions for the Common Core State Standards* 2019, 16-18)

	Result Unknown	Change Unknown	Start Unknown
Add to	*A* bunnies sat on the grass. *B* more bunnies hopped there. How many bunnies are on the grass now? *A* + *B* = ?	*A* bunnies were sitting on the grass. Some more bunnies hopped there. Then there were *C* bunnies. How many bunnies hopped over to the first *A* bunnies? *A* + ? = *C*	Some bunnies were sitting on the grass. *B* more bunnies hopped there. Then there were *C* bunnies. How many bunnies were on the grass before? ? + *B* = *C*
Take From	*C* apples were on the table. I ate *B* apples. How many apples are on the table now? *C* - *B* = ?	*C* apples were on the table. I ate some apples. Then there were *A* apples. How many apples did I eat? *C* - ? = *A*	Some apples were on the table. I ate *B* apples. Then there were *A* apples. How many apples were on the table before? ? - *B* = *A*
	Total Unknown	**Both Addends Unknown**	**Addend Unknown**
Put Together/ Take Apart	*A* red apples and *B* green apples are on the table. How many apples are on the table? *A* + *B* = ?	Grandma has *C* flowers. How many can she put in her red vase and how many in her blue vase? *C* = ? + ?	*C* apples are on the table. *A* are red and the rest are green. How many apples are green? *A* + ? = *C* *C* - *A* = ?

NOTE: This table is is continued on next page.

	Difference Unknown	Bigger Unknown	Smaller Unknown
Additive Compare	"*How many more?*" version. Lucy has *A* apples. Julie has *C* apples. How many more apples does Julie have than Lucy? "*How many fewer*?" version. Lucy has *A* apples. Julie has *C* apples. How many fewer apples does Lucy have than Julie? $A + ? = C$ $C - A = ?$	*"More"* version suggests operation. Julie has *B* more apples than Lucy. Lucy has *A* apples. How many apples does Julie have? "*Fewer*" version suggests wrong operation. Lucy has *B* fewer apples than Julie. Lucy has *A* apples. How many apples does Julie have? $A + B = ?$	"*Fewer*" version suggests operation. Lucy has *B* fewer apples than Julie. Julie has *C* apples. How many apples does Lucy have? "*More*" suggests wrong operation. Julie has *B* more apples than Lucy. Julie has *C* apples. How many apples does Lucy have? $C - B = ?$ $? + B = C$

- In each type of problem, shown as a row, any one of the three quantities in the situation can be unknown, leading to the specific problem situations shown in each cell of the row. The table also shows some important language variations which, while mathematically the same, require separate attention.
- Darker shading indicates the four Kindergarten problem situations. Grade 1 and 2 students work with all problem situations and variations. Unshaded (white) problems illustrate the four problem situations or variations that students should work with in Grade 1 but need not master until Grade 2.
- Put Together/Take Apart Both Addends Unknown problem situations can be used to show all decompositions of a given number, especially important for numbers within 10. Equations with totals on the left help children understand that = does not always mean "makes" or "results in" but always means "is the same amount as." Such problems are not a problem situation with one unknown, as is the Addend Unknown problem situation to the right. These problems are a productive variation with two unknowns that give experience with finding all of the decompositions of a number and reflecting on the patterns involved.
- In Put Together/Take Apart Start Unknown problem situations either addend can be unknown. Both variations should be included.
- For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using more for the bigger unknown and using less for the smaller unknown). The other versions are more difficult.

Table 2A. Multiplication and Division Problem Situations

(Adapted *from Progressions for the Common Core State Standards* 2019, 32)

	A x B = ?	A x ? = C and C ÷ A = ?	? x B = C and C ÷ B = ?
Equal Groups of Objects	**Unknown Product** There are *A* bags with *B* plums in each bag. How many plums are there in all?	**Group Size Unknown** If *C* plums are shared equally into *A* bags, then how many plums will be in each bag?	**Number of Groups Unknown** If *C* plums are to be packed *B* to a bag, then how many bags are needed?
Arrays of Objects	**Equal Groups Language**		
	Unknown Product There are *A* rows of apples with *B* apples in each row. How many apples are there?	**Unknown Factor** If *C* apples are arranged into *A* equal rows, how many apples will be in each row?	**Unknown Factor** If *C* apples are arranged into equal rows of *B* apples, how many rows will there be?
	Rows and Columns Language		
	Unknown Product The apples in the grocery window are in *A* rows and *B* columns. How many apples are there?	**Unknown Factor** If *C* apples are arranged into an array with *A* rows, how many columns of apples are there?	**Unknown Factor** If *C* apples are arranged into an array with *B* columns, how many rows are there?

NOTE: This table is continued on next page.

Multiplicative Compare	**A > 1**		
	Larger Unknown A blue hat costs *$B*. A red hat costs *A* times as much as the blue hat. How much does the red hat cost?	**Smaller Unknown** A red hat costs *$C* and that is *A* times as much as a blue hat costs. How much does a blue hat cost?	**Multiplier Unknown** A red hat costs *$C* and a blue hat costs *$B*. How many times as much does the red hat cost as the blue hat?
	A < 1		
	Smaller Unknown A blue hat costs *$B*. A red hat costs *A* as much as the blue hat. How much does the red hat cost?	**Larger Unknown** A red hat costs *$C* and that is *A* of the cost of a blue hat. How much does a blue hat cost?	**Multiplier Unknown** A red hat costs *$C* and a blue hat costs *$B*. What fraction of the cost of the blue hat is the cost of the red hat?

- Shading indicates the Equal Groups and Array problem begun in Grade 3. Unshaded (white) problems illustrate the Multiplicative Compare problems that appear first in Grade 4, with whole-number values for A, B, and C, and with the "times as much" language in the table. In Grade 5, unit fractions language such as "one third as much" may be used with these Multiplicative Compare problems. Multiplying and unit fraction language change the subject of the comparing sentence, e.g., "A red hat costs A times as much as the blue hat" results in the same comparison as "A blue hat costs 1/A times as much as the red hat," but has a different subject.
- Equal Groups problems can also be stated in terms of columns, exchanging the order of A and B, so that the same array is described. For example: There are B columns of apples with A apples in each column. How many apples are there?
- In the row and column situations, the number of groups and group size are not distinguished.
- Division problems of the form A x ? = C are about finding an unknown multiplicand. For Equal Groups and Compare situations, these involve what is called the sharing, partitive, how-many-in-each-group, or what-is-the-unit interpretation of division. Array situations can be seen as Equal Groups situations, thus, also as examples of the sharing interpretation of division for problems about finding an unknown multiplicand.
- Division problems of the form ? x B = C are about finding an unknown multiplier. For Equal Groups and Compare situations, these involve what is called the measurement, quotitive, how-many-groups, or how-many-units interpretation of division. Array situations can be seen as Equal Groups situations, thus, also as examples of the measurement interpretation of division for problems about finding an unknown multiplier.

Table 2B. Multiplication and Division - Measurement Examples

(Adapted from Progressions for the Common Core State Standards 2019, 103)

	A x B = ?	**A x ? = C and C ÷ A = ?**	**? x B = C and C ÷ B = ?**
Grouped Objects (Units of Units)	You need *A* lengths of string, each *B* inches long. How much string will you need altogether?	You have *C* inches of string, which you will cut into *A* equal pieces. How long will each piece of string be?	You have *C* inches of string, which you will cut into pieces that are *B* inches long. How many pieces of string will you have?
Arrays of Objects (Spatial Structuring)	What is the area of a *A* cm by *B* cm rectangle?	A rectangle has area *C* square centimeters. If one side is *A* cm long, how long is a side next to it?	A rectangle has area *C* square centimeters. If one side is *B* cm long, how long is a side next to it?
Multiplicative Compare	A rubber band is *B* cm long. How long will the rubber band be when it is stretched to be *A* times as long?	A rubber band is stretched to be *C* cm long and that is *A* times as long as it was at first. How long was the rubber band at first?	A rubber band was *B* cm long at first. Now it is stretched to be *C* cm long. How many times as long is the rubber band now as it was at first?

- Shading indicates the Grouped and Array problems begun in Grade 3. Unshaded (white) problems illustrate the Multiplicative Compare problems that appear first in Grade 4.
- In the second column, division problems of the form *A x ? = C* are about finding an unknown multiplicand. For Grouped and Compare situations, these involve what is called the sharing, partitive, how-many-in-each-group, or what-is-the-unit interpretation of division. Array situations can be seen as Equal Groups situations, thus, the Array situations in this column can also be seen as examples of the sharing interpretation of division.

- In the third column, division problems of the form *?* x *B* = *C* are about finding an unknown multiplier. For Equal Groups and Compare situations, these involve what is called the measurement, quotitive, how-many-groups, or how-many-units interpretation of division. Array situations can be seen as Equal Groups situations, thus, the Array situations in this column can also be seen as examples of the measurement interpretation of division.

Table 3. The Properties of Operations.

Here a, b, and c stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

Associative property of addition	$(a + b) + c = a + (b + c)$
Commutative property of addition	$a + b = b + a$
Additive identity property of 0	$a + 0 = 0 + a = a$
Existence of additive inverses	For every a there exists $-a$ so that $a + (-a) = (-a) + a = 0$
Associative property of multiplication	$(a \times b) \times c = a \times (b \times c)$
Commutative property of multiplication	$a \times b = b \times a$
Multiplicative identity property of 1	$a \times 1 = 1 \times a = a$
Existence of multiplicative inverses	For every $a \neq 0$ there exists $1/a$ so that $a \times 1/a = 1/a \times a = 1$
Distributive property of multiplication over addition	$a \times (b + c) = a \times b + a \times c$

Table 4. The Properties of Equality.

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$
Addition property of equality	If $a = b$, then $a + c = b + c$
Subtraction property of equality	If $a = b$, then $a - c = b - c$
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.

Table 5. The Properties of Inequality.

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational or real number systems.

Exactly one of the following is true: $a < b$, $a = b$, $a > b$
If $a > b$, then $b < a$
If $a > b$, then $-a < b$
If $a > b$, then $a \pm c > b \pm c$
If $a > b$ and $c > 0$, then $a \times c > b \times c$
If $a > b$ and $c < 0$, then $a \times c < b \times c$
If $a > b$ and $c > 0$, then $a \div c > b \div c$
If $a > b$ and $c < 0$, then $a \div c < b \div c$

Appendix 2

Glossary

"Children build their own "working definitions" based on their initial experiences … the concepts will become more precise, and the vocabulary with which we name the concepts will, accordingly, carry more precise meanings. Formal definitions generally come last" (Education Development Center 2020).

Appendix 2. Glossary

Addition and subtraction within 5, 10, 20, 100, or 1,000. Addition or subtraction of two whole numbers with whole number answers, and with sum or minuend in the range 0-5, 0-10, 0-20, or 0-100, respectively. Example: 8 + 2 = 10 is an addition within 10, 14 – 5 = 9 is a subtraction within 20, and 55 – 18 = 37 is a subtraction within 100.

Additive inverses. Two numbers whose sum is 0 are additive inverses of one another. Example: 3/4 and – 3/4 are additive inverses of one another because 3/4 + (- 3/4) = (- 3/4) + 3/4 = 0.

Area model. A rectangular shaped tool for understanding problems that involve multiplicative reasoning. An area model may be used to develop conceptual understanding, promote sense making, and build connections between topics.

Associative property of addition. See Table 3 in the Appendix.

Associative property of multiplication. See Table 3 in the Appendix.

Bivariate data. Pairs of linked numerical observations. Example: a list of heights and weights for each player on a football team.

Box plot. A method of visually displaying a distribution of data values by using the median, quartiles, and extremes of the data set. A box shows the middle 50% of the data.

Cardinality. The last number word said when counting tells how many objects have been counted.

Commutative property. See Table 3 in the Appendix.

Complex fraction. A fraction *A/B* where *A* and/or *B* are fractions (*B* nonzero).

Computation algorithm. A set of predefined steps applicable to a class of problems that gives the correct result in every case when the steps are carried out correctly. *See also:* computation strategy.

Computation strategy. Purposeful manipulations that may be chosen for specific problems, may not have a fixed order, and may be aimed at converting one problem into another. *See also:* computation algorithm.

Conceptual subitizing. Recognizing that a collection of objects is composed of subcollections and quickly combining their cardinalities to find the cardinality of the collection (e.g., seeing a set as two subsets of cardinality 2 and saying “four”).

Congruent. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations).

Counting on. A strategy for finding the number of objects in a group without having to count every member of the group. For example, if a stack of books is known to have 8 books and 3 more books are added to the top, it is not necessary to count the stack all over again. One can find the total by counting on—pointing to the top book and saying "eight," following this with "nine, ten, eleven. There are eleven books now."

Dot plot. *See*: line plot.

Dilation. A transformation that moves each point along the ray through the point emanating from a fixed center and multiplies distances from the center by a common scale factor.

Efficiently. In a way that uses strategic thinking to carry out a computation or apply procedures.

Expanded form. A multi-digit number is expressed in expanded form when it is written as a sum of single-digit multiples of powers of ten. For example, 643 = 600 + 40 + 3.

Expected value. For a random variable, the weighted average of its possible values, with weights given by their respective probabilities.

First quartile. For a data set with median M, the first quartile is the median of the data values less than M. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the first quartile is 6. *See also:* median, third quartile, interquartile range.

Flexibly. In a way that can use, explain, and justify multiple strategies and be adept at choosing a meaningful strategy for the computation or procedure in a problem.

Fraction. A number expressible in the form *a/b* where *a* is a whole number and b is a positive whole number. (The word fraction in these standards always refers to a non-negative number.) *See also:* rational number.

Hierarchical inclusion. Each whole number represents one more than the previous number in the counting sequence and includes all previous numbers within it.

Identity property of 0. See Table 3 in the Appendix.

Independently combined probability models. Two probability models are said to be combined independently if the probability of each ordered pair in the combined model equals the product of the original probabilities of the two individual outcomes in the ordered pair.

Integer. A number expressible in the form *a* or -*a* for some whole number *a*.

Interquartile range. A measure of variation in a set of numerical data, the interquartile range is the distance between the first and third quartiles of the data set. Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the interquartile range is 15 – 6 = 9. *See also:* first quartile, third quartile.

Line plot. A method of visually displaying a distribution of data values where each data value is shown as a dot or mark above a number line. Also known as a dot plot.

Mathematical modeling. A process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena.

Mathematize. To treat or regard mathematically.

Mean. A measure of center in a set of numerical data, computed by adding the values in a list and then dividing by the number of values in the list.[4] Example: For the data set {1, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean is 21.

Mean absolute deviation. A measure of variation in a set of numerical data, computed by adding the distances between each data value and the mean, then dividing by the number of data values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the mean absolute deviation is 20.

Measurement division. Division when finding the number of groups (an unknown multiplier) and of the form *? x B = C* . This type of division can also be referred to as quotitive, how-many-groups, or how-many-units interpretation of division. Also called quotitive division.

Median. A measure of center in a set of numerical data. The median of a list of values is the value appearing at the center of a sorted version of the list—or the mean of the two central values, if the list contains an even number of values. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 90}, the median is 11.

Midline. In the graph of a trigonometric function, the horizontal line halfway between its maximum and minimum values.

Multiplication and division within 100. Multiplication or division of two whole numbers with whole number answers, and with product or dividend in the range 0-100. Example: 72 ÷ 8 = 9.

Multiplicative inverses. Two numbers whose product is 1 are multiplicative inverses of one another. Example: 3/4 and 4/3 are multiplicative inverses of one another because 3/4 x 4/3 = 4/3 x 3/4 = 1.

Number conservation. Understanding that the quantity of a set doesn't change if the set is rearranged.

Number line. Used to represent numbers and support reasoning about them.

One to one correspondence. Saying number words in correspondence with the objects counted.

Partitive division. Division when finding the size of each group (an unknown multiplicand) and of the form A x ? = C. This type of division can also be referred to as the sharing, how-many-in-each-group, or what-is-the-unit interpretation of division. *See also* sharing division

Percent rate of change. A rate of change expressed as a percent. Example: if a population grows from 50 to 55 in a year, it grows by 5/50 = 10% per year.

Perceptual subitizing. Quickly recognizing the cardinalities of small groups without having to count the objects.

Probability distribution. The set of possible values of a random variable with a probability assigned to each.

Properties of equality. See Table 4 in the Appendix.

Properties of inequality. See Table 5 in the Appendix.

Properties of operations. See Table 3 in the Appendix.

Probability. A number between 0 and 1 used to quantify likelihood for processes that have uncertain outcomes (such as tossing a coin, selecting a person at random from a group of people, tossing a ball at a target, or testing for a medical condition).

Probability model. A probability model is used to assign probabilities to outcomes of a chance process by examining the nature of the process. The set of all outcomes is called the sample space, and their probabilities sum to 1. *See also:* uniform probability model.

Quotitive division. Division when finding the number of groups (an unknown multiplier) and of the form $? \times B = C$. This type of division can also be referred to as the measurement, how-many-groups, or how-many-units interpretation of division. Also called measurement division.

Random variable. An assignment of a numerical value to each outcome in a sample space.

Rational expression. A quotient of two polynomials with a non-zero denominator.

Rational number. A number expressible in the form a/b or $-a/b$ for some fraction a/b. The rational numbers include the integers.

Rectilinear figure. A polygon all angles of which are right angles.

Rekenrek. Developed by mathematics education researchers at the Freudenthal Institute in the Netherlands. A visual model comprised of two strings of ten beads each, strategically broken into two groups: five red beads, and five white beads. Also known as an arithmetic rack or math rack.

Rigid motion. A transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations. Rigid motions are here assumed to preserve distances and angle measures.

Repeating decimal. The decimal form of a rational number. *See also:* terminating decimal.

Sample space. In a probability model for a random process, a list of the individual outcomes that are to be considered.

Scatter plot. A graph in the coordinate plane representing a set of bivariate data. For example, the heights and weights of a group of people could be displayed on a scatter plot.

Sharing division. Division when finding the size of each group (an unknown multiplicand) and of the form $A \times ? = C$. This type of division can also be referred to as partitive, how-many-in-each-group, or what-is-the-unit interpretation of division. Also called partitive division.

Similarity transformation. A rigid motion followed by a dilation.

Tape diagram. A drawing that looks like a segment of tape, used to illustrate number relationships. Also known as a strip diagram, bar model, fraction strip, or length model.

Terminating decimal. A decimal is called terminating if its repeating digit is 0.

Third quartile. For a data set with median *M*, the third quartile is the median of the data values greater than *M*. Example: For the data set {2, 3, 6, 7, 10, 12, 14, 15, 22, 120}, the third quartile is 15. *See also:* median, first quartile, interquartile range.

Transitivity principle for indirect measurement. If the length of object A is greater than the length of object *B*, and the length of object *B* is greater than the length of object C, then the length of object *A* is greater than the length of object *C*. This principle applies to measurement of other quantities as well.

Uniform probability model. A probability model which assigns equal probability to all outcomes. *See also:* probability model.

Vector. A quantity with magnitude and direction in the plane or in space, defined by an ordered pair or triple of real numbers.

Visual fraction model. A tape diagram, number line, or area model.

Whole numbers. The numbers 0, 1, 2, 3, …

Appendix 3

Wisconsin's Shifts in Mathematics

"Exploration and understanding are at the heart of what it means to do mathematics" (Su 2020, 23).

Appendix 3. Wisconsin's Shifts in Mathematics, 2021

The 2021 *Wisconsin Standards for Mathematics* are built on the foundation of existing standards (Council of Chief State School Officers 2010) and reflect new research and understandings of mathematics. Wisconsin's standards identify the knowledge, skills, and habits that will allow students to succeed in their chosen paths. Understanding how the standards differ from previous standards and how Wisconsin standards differ from national standards is essential to implementing Wisconsin's standards well and selecting, adopting, and personalizing standards-aligned instructional materials.

There are five important shifts from previous standards to the 2021 *Wisconsin Standards for Mathematics*. Identifying the shifts builds understanding of how these standards differ from previous standards. The shifts also serve as a tool that educators can use to identify what is necessary in standards-aligned instruction and assessment at a high level. Three of the five shifts are from the 2010 standards (Council of Chief State School Officers 2010) but have been expanded upon to emphasize advancing educational equity in mathematics. Two of the five shifts are new and unique to Wisconsin.

The following are shifts in the 2021 *Wisconsin Standards for Mathematics*:

Shift #1: Learning mathematics emphasizes recognizing, valuing, and fostering mathematical identities and agency in all students.

The *Wisconsin Standards for Mathematics* (2021) expects opportunities for inclusion of broader ways to think and do mathematics. This shift supports recognizing and valuing the mathematical ways of thinking students bring with them to school mathematics from their culture, their families, or previous grade level. By leveraging multiple mathematical competencies, drawing on multiple resources of knowledge, and going deep into the mathematical concepts, students develop stronger mathematical understanding (Aguirre, Mayfield-Ingram, and Martin 2013, 43).

Instruction and instructional materials for mathematics promote fostering mathematical identities and agency by reinforcing that:

- "How students are positioned to participate in mathematics affects not only what they learn but also how they come to see themselves as learners. The ways in which students view themselves as learners of mathematics greatly influence how they participate" (NCTM 2018, 28).
- "A positive productive mathematics learner identity contributes to the mathematical learning of a child. Instruction that values multiple mathematical contributions, provides multiple entry points and promotes student participation

in various ways (teams, groups, and so on) can aid the development of a student's mathematical learning identity" (Aguirre, Mayfield-Ingram, and Martin 2013, 43).

- "Actively working to notice and disrupt societal views of what it looks like to be "good at math" is essential for seeing all students' mathematical strengths and for nurturing positive mathematics identities for all students" (Skinner, Louie, and Baldinger 2019, 340).
- "Math is the creation of people—people working together and depending on one another. Interaction, cooperation, and group communications, therefore, are key components of this process. Students also help generate part of the content of instruction as well. They participate in the physical event that will generate data which becomes the vehicle for introducing mathematical concepts. Cooperation and participation in group activities, as well as personal responsibility for individual work, become important not only for the successful functioning of the learning group, but for the generation of instructional material and various representations of the data as well" (Moses and Cobb 2001, 120).
- "Successful math users...approach math with the desire to understand it and to think about it, and with the confidence that they can make sense of it. Successful users of math search for patterns and relationships and think about connections. They approach math with a *mathematical mindset*, knowing that math is a subject of growth and that their role is to learn and think about new ideas" (Boaler 2018).
- "Mathematics becomes joyful when children have opportunities to learn mathematics in ways they see as relevant to their identities and communities and when they are encouraged to explore, create, and make meaning in mathematics" (NCTM 2020, 18).

Shift #2: All students are flexible users of mathematics who see how mathematics can be used to understand their world and the world around them.

The *Wisconsin Standards for Mathematics* (2021) call for empowering students to be the thinkers and doers of mathematics. The standards are calling for an intentional pairing of the Standards for Mathematical Practice and the Standards for Mathematical Content that allow for students to gain a lifelong appreciation of mathematics and how mathematics is used to critique and understand the world.

Mathematical modeling is one powerful way to bring this shift to life in students' mathematical journeys. Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions, or otherwise provide insight into real-world phenomena (Bliss and Libertini 2016, 8). See Appendix 4, Mathematical Modeling for more details about the K-12 modeling process.

Instruction and instructional materials for mathematics promote students as flexible users of mathematics by reinforcing that:

- "A math memorizer doesn't know how to react in unfamiliar situations, but a math explorer can flexibly adapt to changing conditions, because she has learned to ask questions that will prepare her for many scenarios" (Su 2020, 26).
- "As children gain familiarity with representations, they become tools for problem solving. Children can then draw on different representations as resources to access and make sense of problem situations and the underlying mathematical relationships. Accessing multiple resources positions children as competent" (NCTM 2020, 64).
- "Mathematics is something people *do* rather than something disconnected from human experience. When we see mathematics as something one *does*, we see the potential of mathematics as a tool to understand, critique, and create change in the world" (NCTM 2020, 15).
- "It is impossible for students to engage in modeling with mathematics without engaging in other math practices. At the very least, students should also be engaged in making sense of problems and persevering in solving them, reasoning abstractly and quantitatively, and attending to precision. As tasks get more complex, students should also be engaged in constructing viable arguments and critiquing the reasoning of others, using appropriate tools strategically, and looking for and making use of structure" (Galasso 2016).
- "Student opinions matter and influence their answer to a question. They still have to do the same mathematics to answer the question, but they are forced to reconcile their answer with reality, making the mathematics more relevant and interesting. Making judgments about what matters and assessing the quality of a solution are components of mathematical modeling" (Bliss and Libertini 2016,10).
- "Solving problems as a computational thinker also requires some specific attitudes toward problem solving in general. We strive to empower our students with the confidence needed to tackle ambiguous problems, the tenacity to persist through challenges requiring iteration and experimentation, strong communication skills to facilitate collaboration and presentation, and a general curiosity across all disciplines that leads them to ask and answer big, scary questions" (Sheldon 2017).

Shift #3: All students engage in mathematics that is focused on developing deep understanding of and connections among mathematical concepts in order to gain strong foundations to move their mathematics learning forward.
(This 2010 shift, included with the release of the *Common Core State Standards* in 2010, is still applicable and supported in the 2021 *Wisconsin Academic Standards for Mathematics*.)

The *Wisconsin Standards for Mathematics* (2021) continue to call for a focus in mathematics content at each grade level. The standards significantly narrow and deepen the way time and energy are spent. This means focusing deeply on the major work of each grade as follows:

- In grades K–2: Concepts, skills, and problem solving related to addition and subtraction
- In grades 3–5: Concepts, skills, and problem solving related to multiplication and division of whole numbers and fractions
- In grade 6: Ratios and proportional relationships, and early algebraic expressions and equations
- In grade 7: Ratios and proportional relationships, and arithmetic of rational numbers
- In grade 8: Linear algebra and linear functions

This focus will help students gain strong foundations, including a solid understanding of concepts to support their mathematical experiences. Students develop a strong foundational knowledge and deep conceptual understanding and are able to transfer mathematical skills and understanding across concepts and grades (NMAP 2008, 15-20).

Instruction and instructional materials for mathematics promote students engaging in focused mathematics by reinforcing that:

- "Focus is necessary in order to fulfill the ambitious promise the states have made to their students by adopting the Standards: greater achievement at the college and career ready level; greater depth of understanding of mathematics; and a rich classroom environment in which reasoning, sense-making, applications, and a range of mathematical practices flourish" (Students Achievement 2015, 7).
- "Developing deep understanding of mathematics is a major goal of equity-based mathematics teaching practices" (Aguirre, Martin, and Mayfield-Ingram 2013, 43).

- "Teachers and children need access to high-quality mathematics curriculum materials and dedicated time for mathematics instruction in order to press for deep conceptual understanding of mathematics" (NCTM 2020, 78).
- "Conceptual understanding is not like an on-off light switch: You don't *understand* a concept in an all-or-nothing fashion. Initially we grasp some aspect of the concept and build upon it, adding and elaborating our understanding...In general, the more connections of the right kind, the more examples in different but relevant contexts, the more elaborate the networks of ideas and relationships—the deeper, richer, more generalized, and more abstract is our understanding of a concept" (Hyde 2006, 41).
- "The collaborative focus on mathematics learning among the adults in the educational setting can then serve as a catalyst for creating a school culture of deeper mathematics learning and mathematically powerful learning spaces for each and every child" (NCTM 2020, 121).
- "...the high school mathematics curriculum should help students see the big picture, the key concepts, and the connections between and among the concepts that they are learning" (NCTM 2018, 38).

Shift #4: All students engage in coherent mathematics that connects concepts and mathematical thinking within and across domains and grades.
(This 2010 shift, included with the release of the *Common Core State Standards* in 2010, is still applicable and supported in the 2021 *Wisconsin Academic Standards for Mathematics*.)

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts. Therefore, the standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years. Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics.

Instruction and instructional materials for mathematics promote students engaging in coherent mathematics by reinforcing that:

- "Each standard is not a new event, but an extension of previous learning" (Common Core State Standards Initiative 2010).

- “It’s about math making sense. The power and elegance of math comes out through carefully laid progressions and connections within grades” (Student Achievement Partners 2016).

- “Equity-based teaching depends on the capacity to recognize and intentionally tap students’ knowledge and experiences—mathematical, cultural, linguistic, peer, family, and community—as resources for mathematics teaching and learning. Drawing on this knowledge and experience includes helping students bridge everyday experiences to learn mathematics, capitalizing on linguistic resources to support mathematics learning, recognizing family or community mathematical practices to support mathematics learning, and finding ways to help students learn and use mathematics to solve authentic problems that affect their lives” (Aguirre, Mayfield-Ingram, and Martin 2013, 43-44).

- “When students grapple with something new, they often try to use concepts they’ve figured out in other contexts to see whether they’ll work under the new conditions. They might extend mathematics they’ve learned in school or connect to experiences they’ve had in their daily lives” (Zager 2017, 200).

Shift #5: All students engage in rigorous mathematics focused with equal intensity on developing conceptual understanding, procedural flexibility and efficiency, and application to authentic contexts.
(This 2010 shift, included with the release of the *Common Core State Standards* in 2010, is still applicable and supported in the 2021 *Wisconsin Academic Standards for Mathematics*.)

Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, flexible and efficient procedural skills, and application.

Conceptual understanding: The standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.

Flexible and efficient procedural skills: The standards call for efficiency and accuracy in calculation. Flexible and efficient procedural skills build from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems (NCTM 2014).

Application: The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural flexibility and efficiency.

Instruction and instructional materials for mathematics promote students engaging in rigorous mathematics by reinforcing that:

- "Rigor refers to an integrated way in which conceptual understanding, strategies for problem solving and computation, and applications are learned, so that each supports the other" (California Department of Education 2021, 27).
- "Students with positive mathematics identities and strong mathematical agency—the capacity and willingness to engage mathematically—are often those students who have deep conceptual understanding of mathematical ideas, relationships, and operations. These students are able to use mathematical procedures meaningfully because they understand why they work and when it is appropriate to use them" (Huinker and Bill 2017, 68).
- "When knowledge of procedures builds from conceptual understanding, students see mathematics as making sense and are able to use mathematical procedures meaningfully and appropriately to solve contextual and mathematical problems" (Huinker and Bill 2017, 67).

Appendix 4

Mathematical Modeling

"...one of the most important uses of modeling in the classroom is for the experience of creating a model all one's own" (Godbold, Malkevitch, Teague, and van der Kooij 2016, 54).

Appendix 4. Mathematical Modeling

Providing students with regular mathematical modeling experiences from kindergarten through grade 12 is one important way that educators can support students as flexible users of mathematics who see how mathematics can be used to understand their world and the world around them (WI Shift #2). See Appendix 3, *Wisconsin's Shifts in Mathematics, 2021* for more details about Shift #2 and how this shift can serve as a tool that educators can use to identify what is necessary in standards-aligned instruction and assessment at a high level.

This appendix aims to provide multiple connections between equitable teaching and mathematical modeling, describe the modeling process, highlight the interdisciplinary opportunities that are present, provide a starting point for developing a K-12 progression of modeling within WI schools and districts, and a few thoughts about assessment as it relates to mathematical modeling.

Equitable Teaching and Mathematical Modeling

Equitable teaching in mathematics can be supported by reflecting on three questions: What mathematics? For whom? For what purposes? Meaningful reflection on these questions asks teachers to examine their beliefs about learners, learning, mathematical content, and their everyday teaching practices (Aguirre, Mayfield-Ingram, and Martin 2013, 5).

Drawing on students' funds of knowledge, a research-based equitable mathematics teaching practice, has strong connections to the Standards for Mathematical Practice (SMP), including SMP 4: Model with mathematics. "Mathematical modeling draws on students' funds of knowledge by situating the problem posing in the local community, school, and students' lived experiences. Assumptions are drawn based on experiences and ideas draw from resources in their community" (Suh 2020). This relationship between mathematical modeling and the local community provides a space where school becomes a place to sustain one's cultural identity by sharing community and cultural math happenings. When problem posing is strongly connected to students' lived experiences, strong feelings may surface. Teachers are encouraged to establish supportive classroom norms regarding discourse so that students can safely share their thoughts and ideas within their learning community (Suh 2020).

Mathematical modeling provides opportunities to develop students' appreciation for the role mathematics plays in real life and provides "multiple points of assessment for teachers to look for and amplify the strength in student thinking" (Suh 2020). "Mathematical modeling has an asset-based pedagogy in place to celebrate each students' contribution and differentiate naturally in the mathematical modeling process" (Suh 2020). "The open-ended aspect of mathematical modeling, the variety of approaches to a problem, the need to present and interpret the ideas behind a model clearly, and the need to apply modeling in a variety of real-world situations all combine to create opportunities for individual students to contribute to a group's work in many ways" (Godbold, Malkevitch, Teague, and van der Kooij 2016, 57).

"Mathematical modeling is a rich discursive process that includes students using multiple representations such as visuals, numbers, and words to express one's mathematical thinking" (Suh 2020). As students analyze their model throughout the process, students exchange ideas while refining the model and advance their thinking. These student-to-student exchanges provide opportunities for modelers to appreciate the differing perspectives that students bring to those discussions, supporting a strong modeling process.

Mathematical modeling supports a broadening of what it means to think and do mathematics. The modeling process provides experiences to show that not all students will use the same model as they grapple with the same question and there is not always only one answer to a problem. Modeling activities can help avoid misconceptions that a speedy solution is best. Mathematical modeling provides students with more sophisticated ways to determine the validity and usefulness of a problem-solving strategy or solution or may provide motivation for new tools (Galluzzo, Levy, Long, and Zbiek 2016, 24). Ultimately, creating a model all one's own can be exciting and transformational for students. "Math modeling doesn't just change their relationship with mathematics, but their view of themselves and their relationship with mathematics" (Godbold, Malkevitch, Teague, and van der Kooij 2016, 54).

The Mathematical Modeling Process

"Mathematical modeling provides authentic connections to real-life situations. The process starts with ill-defined, often messy real-life problems that may have several different solutions that are all correct. Mathematical modeling requires the modeler to be critical and creative as well as make choices, assumptions, and decisions. Through this process, students create a mathematical model that describes a situation using mathematical concepts and language. The model can be used to solve a problem or make decisions and can be used to deepen understanding of mathematical concepts.

The process of mathematical modeling has key components that are interconnected and applied in an iterative way. Students may move between and across, as well as return to each of the components as they change conditions to observe new outcomes until the model is ready to be shared and acted upon" (Ontario Curriculum and Resources 2020). While moving through these components, social-emotional learning skills, as they relate to the Standards for Mathematical Practice, are applied as needed. Moves toward both a positive mathematical identity and a strong sense of agency as a thinker and doer of mathematics are desired outcomes stemming from these student experiences with the mathematical modeling process (Ontario Curriculum and Resources 2020).

Content Connections

Tasks that require the process of mathematical modeling can be strategically situated throughout the year to support students in making connections among mathematical concepts, mathematical domains, other content areas, and provide opportunities for assessing the integration and application of learning (Ontario Curriculum and Resources 2020).

While the idea of launching mathematical modeling problems from questions that children naturally ask may seem to align to the sciences most easily, modeling can also tie mathematical tools to other content areas including social studies, literacy, art, physical education, and music. By specifically "encouraging children to explore naturally arising quantitative questions using mathematics tools," mathematical modeling "can strengthen their understanding of academic material and simultaneously reinforce the need for math" (Galluzzo, Levy, Long, and Zbiek 2016, 25).

One example of modeling as a process for making connections with other content areas is modeling's strong connection to the practice of science. *Wisconsin Standards for Science* includes two Science and Engineering Practices (SEP) that articulate the use of models, mathematics, and computational thinking to make sense of and solve problems:

- Standard SCI.SEP2: Students develop and use models, in conjunction with using crosscutting concepts and disciplinary core ideas, to make sense of phenomena and solve problems.
- Standard SCI.SEP5: Students use mathematics and computational thinking, in conjunction with using crosscutting concepts and disciplinary core ideas, to make sense of phenomena and solve problems.

While SEP2 is broader than MP4, "encompassing both mathematical modeling and other relevant forms of modeling in science, the importance of mathematical modeling and mathematical models is emphasized in the description of SEP2" (Cirillo, Felton-Koestler, Pelesko, and Rubel 2016, 13). SEP5 highlights a strong connection among science, mathematics, and computational thinking. Computational thinking is a set of overlapping problem solving skills, which can be used in a variety of different settings and therefore shares some of the important aspects of the modeling process. Computational thinking emphasizes the importance of reframing a problem so that it can be represented by a model, considering and testing many possible models before selecting one to implement, and understanding that models are representational so there are many ways to understand the problem or improve the model (Digital Promise 2018, 20-21).

A second example of modeling as a process for making connections with other content areas is modeling as it relates to *Wisconsin Standards for Computer Science*. Both Algorithms and Programming as well as Data Analysis within the computer science standards provide such a connection. Many mathematical modeling problems benefit from the use of algorithms or programming, especially when the modeling involves repeated calculations. In these situations, digital computational learning can be presented in parallel with developing mathematical skills and applied where useful (Sanford and Naidu 2016, 25). Since mathematical modeling is based on data and analyzing that data, there are several additional computer science standards that again make strong connections to mathematical modeling. Students recognize and define data in computational problems (Standard CS.DA2), communicate about that data (Standard CS.DA3), and develop and use data abstractions (Standard CS.DA4) as they experience the mathematical modeling process.

Finally, mathematical modeling and statistical thinking have a critical relationship with one another. Statistics exists to offer other fields of study a coherent set of ideas and tools for dealing with data, especially dealing with the variability in data (GAISE II 2020, 6).

The statistical problem-solving process has many similarities to the process of mathematical modeling. The four-step statistical problem-solving process includes formulating a statistical investigative question, collecting or considering the data, analyzing the data, and interpreting the results (GAISE II 2020, 12).

The strategic use of questioning throughout both the statistical problem-solving and mathematical modeling processes is essential and supports each of their interactive natures. It is critical that all individuals can use sound statistical reasoning to make evidence-based decisions as a member of society. Statistical literacy is necessary for employment, being informed about current events, and preparing for a healthy, happy, and productive life (GAISE II 2020, 3).

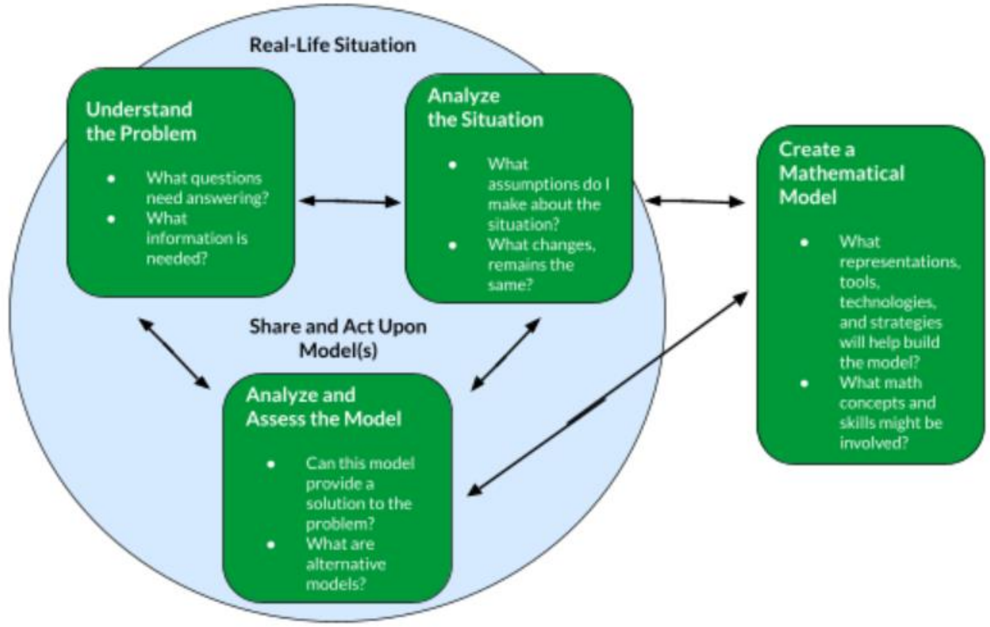
“Mathematical modeling should be taught at every stage of a student’s mathematical education” (Bliss and Libertini 2016, 7). To support Wisconsin teachers and students in their K-12 modeling journeys, the following progression and hallmarks of the process are offered.

Grades K-8

The process of mathematical modeling for kindergarten through grade 8 consists of four components:

- Understand the Problem
- Analyze the Situation
- Create a Mathematical Model
- Analyze and Assess the Model

K-8 Process of Mathematical Modeling



Adapted from Ontario Curriculum and Resources 2020.

Mathematical modeling requires iterating and interconnecting through the four components as shown in the diagram to the right. These are not steps or stages, which suggest a linear path. Instead, it is expected that students will revisit thinking and decisions that were made in previous components. The interplay between the components is required for true mathematical modeling.

This diagram, with its four components, offers several supportive features for teachers of students in kindergarten through grade eight:

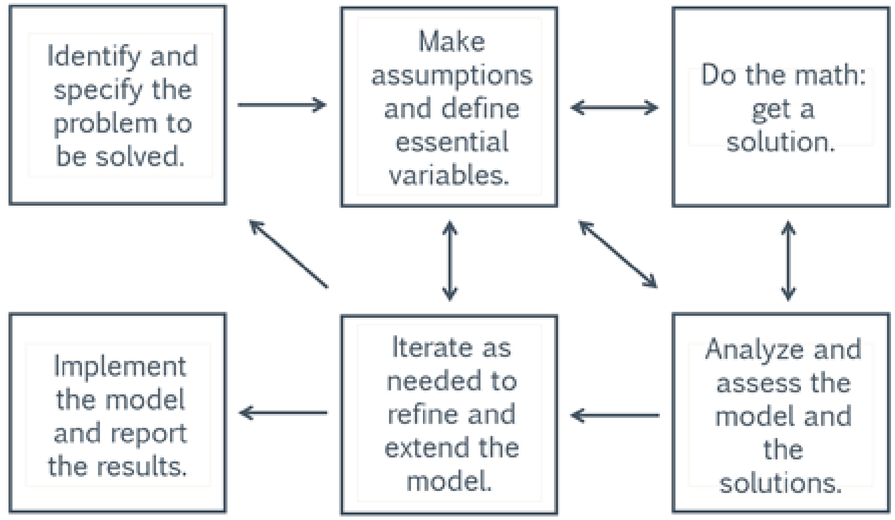
- The diagram includes components that use familiar language to describe the process.
- The K-8 diagram of the modeling process supports our youngest learners and their teachers as they engage in the modeling process as a natural extension of noticing and wondering about situations they come across daily, both in school and outside the classroom. They Understand the Problem, Analyze the Situation, Create a Model, and Analyze and Assess the Model. Then within each of the components, educators and their students dig into what is included in that part of the modeling

process. For example, Analyze the Situation includes making assumptions (or thinking about "what matters") and defining variables (or thinking about what changes and remains the same).

- The visual clearly differentiates the action of mathematical modeling from the product of a mathematical model.
- This diagram shows most of the interactive process of modeling happening within the real-life situation and then the creation of a mathematical model lying outside of that situation, but clearly connected to the process. It is also important to note that mathematical models can often be used outside of mathematical modeling, when they are not part of this iterative process, but instead are used to simply represent or solve a problem.
- The components are not numbered.
- Not numbering the modeling components reinforces that the process is not linear or experienced step by step. Modeling starts with understanding the problem, but after that, students may move back and forth between components as their thinking and findings lead them. This back and forth will continue until the modelers are satisfied that their solution will help their audience.

High School

The diagram for the high school process of mathematical modeling has six components that are interconnected and iterative. As in the K-8 diagram, these are not steps or stages which suggest a linear path. Instead, choices, assumptions, and approximations are present throughout the modeling cycle and the process may have students return to a previous component before going on to complete the cycle.



Source: GAIMME Report, 2016

This diagram, with its six components, offers high school modelers and their teachers a similar process to the K-8 diagram, but with a few aspects of the process that now appear as a separate, more formal component:

- *The diagram becomes more explicit and includes technical language to describe the process.* The high school diagram of the modeling process supports high school learners and their teachers as they engage in the modeling process as an extension of the understandings that were already built during modeling in kindergarten through grade 8. The language used and the results of the modeling process is now more formal. For example, students Make Assumptions and Define Essential Variables and Implement the Model and Report the Results.

- *The visual folds the mathematical model into the interactive modeling process.* High school students are able to readily reason abstractly and quantitatively since they have spent many years understanding relationships among numbers and meanings of operations and how they relate to one another. There is no longer a focus in the diagram on which components of mathematical modeling are within and outside the real-life situation.

- *The components are not numbered.* This is consistent with the K-8 diagram. Not numbering the modeling components reinforces that the process is not linear or experienced step by step. Modeling starts with identifying and specifying the problem to be solved, but after that, students may move back and forth between components as their thinking and findings lead them. This back and forth will continue until the modelers are satisfied that their solution will help their audience, and they are ready to report the results.

The following is a progression of how the components of the mathematical modeling process can develop across grades K-12. The progression has been informed by *Guidelines for Assessment & Instruction in Mathematical Modeling Education* (GAIMME) 2016 and Ontario's *Mathematical Modelling Resources* 2020. It is important for educators to remember that how modeling tasks are organized in the upper grades depends on students' prior experiences with modeling. In addition, each grade band aims to highlight ways of thinking and doing mathematics that may now be available to students as they continue their modeling journey. Aspects of modeling that are outlined in the initial columns of the progression are still valid and present in the subsequent grade bands.

	K-2	**3-5**	**6-8**	**High School**
Understand the Problem	The teacher often chooses a scenario that is relatable for their students and has questions that someone cares about. The teacher considers the mathematical content that has been developed and the tools students have access to when choosing an interesting and accessible setting. The problem can be posed as a story where an explanation, decision, or strategy is needed.	The teacher often presents a rich scenario. Then together, teacher and students refine the question that they will use modeling to answer. Students may begin to participate in the selection and development of problems from their own experiences. There will be times when it will be helpful for students to work on problems that involve small decisions and pieces of the modeling process.	Both teachers and students continue to pose questions that have relevance to their world. While a whole class might be considering a big question, small groups might look at a particular aspect of the problem that interests them. There will still be times when it will be helpful for students to work on problems that involve small decisions and pieces of the modeling process.	Life experiences and interests of students are rapidly expanding, providing an even larger array of real-world problems at their disposal. Problems can be more complex, involve more variables, and offer a wider array of possible analytical and technology-based approaches. There will still be times when it will be helpful for students to work on problems that involve small decisions and pieces of the modeling process.

Note: Continues on next page.

	K-2	3-5	6-8	High School
Understand the Problem (cont'd)	Students may sometimes engage in only part of the modeling cycle, or a modeling context will be used to highlight a particular modeling technique that can be used more creatively in later experiences.			
Analyze the Situation	Students figure out "what matters" and therefore are making assumptions without the formality of modeling vocabulary. Creativity is important in this part of the process since that might inspire innovation. The teacher does help to focus student thinking though, since younger students may generate many options.	Students use the vocabulary of modeling, such as "making assumptions." Students make estimations for some information that is not available and justify those assumptions. Students consider the compatibility or incompatibility of multiple assumptions or implications of their assumptions. Students can begin to consider the idea of variables in the modeling process as they think about what remains constant and what varies.	Students are likely to make more sophisticated arguments, while simultaneously taking into consideration several constraints. Teachers encourage students to be more independent as they attack bigger problems with a growing collection of mathematical tools. A collaborative effort may be supported by individuals taking responsibility for different aspects of the process. Students begin to use dependent and independent variable language.	Teachers consider the modeling experience of their students in case K-8 modeling experiences haven't been fully in place and build in scaffolds with leading questions and class discussions as needed. Visual diagrams, such as mind maps, can be a powerful tool leading to the structure of a model. Students learn how creating specific examples from a context is an important aspect of modeling.

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	K-2	3-5	6-8	High School
Create a Mathematical Model	Students show and play act to demonstrate their ideas. Students communicate or draw simple representations of mathematical ideas.	Students select and apply more sophisticated mathematical tools and visual representations to develop their models. Students consider the predictive potential when constructing the models they build.	Students begin to understand that the value of the model increases if it considers all of the quantities that might vary. The model will be more applicable in different circumstances when users can use their own data within the model. Students begin to use variables to represent quantities that change and use equations to represent relationships between the quantities. This might look like developing a scoring system or a voting method.	Students may devise approaches that will be new to the teacher. This can be both energizing and challenging. Students might utilize a variety of mathematical concepts and skills including those from middle school when engaging with modeling situations. Not every aspect of mathematical modeling may be focused on high school standards.
Analyze and Assess the Model	Students think about their solution in terms of the original question. Does the solution work to answer the mathematical question or should revisions be made?	Students think about how a model might change if different assumptions are made. Students connect their choice of tools to the assumptions and variables.	Students provide multiple representations of their results including symbolic, graphical, and verbal (written and spoken).	Students present and share their work in written and oral form. The feedback received from the teacher and peers is critical in their development as modelers.

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	K-2	3-5	6-8	High School
Analyze and Assess the Model (cont'd)		Students generalize and argue their claims. Students may try to quantify the possible sources of error or uncertainty in a model.	Students can discuss the pros and cons of the choices they made both in the assumptions component and in the choice of mathematical technique. Students may possibly demonstrate and discuss what happens to their solution when they change an assumption or a particular number.	Once satisfied, the model is implemented, and the results are reported to the intended audience.

Assessment

The assessment of mathematical modeling should focus on the process, not simply the product. Mathematical models are very much a part of the assumptions and decisions that were made when working to create them. Assessment should be in service of helping students improve their ability to understand the modeling process and actively model, which will, in time, result in a better product (Bliss and Libertini 2016, 21). Quality modeling includes reflection and communication about the result as well as the model. Identification of mathematical misconceptions and mistakes should be a part of the revision and evaluation processes but is just one of many important factors to be considered in the assessment of mathematical modeling (Galluzzo, Levy, Long, and Zbiek 2016, 42).