

WISCONSIN STANDARDS FOR
Mathematics



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Section I

Wisconsin's Approach to Academic Standards

Purpose of the Document

The purpose of this guide is to improve mathematics education for students and for communities. The Wisconsin Department of Public Instruction (DPI) has developed standards to assist Wisconsin educators and stakeholders in understanding, developing and implementing course offerings and curriculum in school districts across Wisconsin.

This publication provides a vision for student success and follows [The Guiding Principles for Teaching and Learning \(2011\)](#). In brief, the principles are:

1. Every student has the right to learn.
2. Instruction must be rigorous and relevant.
3. Purposeful assessment drives instruction and affects learning.
4. Learning is a collaborative responsibility.
5. Students bring strengths and experiences to learning.
6. Responsive environments engage learners.

Program leaders will find the guide valuable for making decisions about:

- Program structure and integration
- Curriculum redesign
- Staffing and staff development
- Scheduling and student grouping
- Facility organization
- Learning spaces and materials development
- Resource allocation and accountability
- Collaborative work with other units of the school, district, and community

What Are the Academic Standards?

Wisconsin Academic Standards specify what students should know and be able to do in the classroom. They serve as goals for teaching and learning. Setting high standards enables students, parents, educators, and citizens to know what students should have learned at a given point in time. In Wisconsin, all state standards serve as a model. Locally elected school boards adopt academic standards in each subject area to best serve their local communities. We must ensure that all children have equal access to high-quality education programs. Clear statements about what students must know and be able to do are essential in making sure our schools offer opportunities to get the knowledge and skills necessary for success beyond the classroom.

Adopting these standards is voluntary. Districts may use the academic standards as guides for developing local grade-by-grade level curriculum. Implementing standards may require some school districts to upgrade school and district curriculums. This may result in changes in instructional methods and materials, local assessments, and professional development opportunities for the teaching and administrative staff.

What is the Difference between Academic Standards and Curriculum?

Standards are statements about what students should know and be able to do, what they might be asked to do to give evidence of learning, and how well they should be expected to know or do it. Curriculum is the program devised by local school districts used to prepare students to meet standards. It consists of activities and lessons at each grade level, instructional materials, and various instructional techniques. In short, standards define what is to be learned at certain points in time, and from a broad perspective, what performances will be accepted as evidence that the learning has occurred. Curriculum specifies the details of the day-to-day schooling at the local level.

Developing the Academic Standards

DPI has a transparent and comprehensive process for reviewing and revising academic standards. The process begins with a notice of intent to review an academic area with a public comment period. The State Superintendent's Standards Review Council examines those comments and may recommend revision or development of standards in that academic area. The state superintendent authorizes whether or not to pursue a revision or development process. Following this, a state writing committee is formed to work on those standards for all grade levels. That draft is then made available for open review to get feedback from the public, key stakeholders, educators, and the Legislature with further review by the State Superintendent's Standards Review Council. The state superintendent then determines adoption of the standards.

Aligning for Student Success

To build and sustain schools that support every student in achieving success, educators must work together with families, community members, and business partners to connect the most promising practices in the most meaningful contexts. The release of the Wisconsin Standards for Mathematics provides a set of important academic standards for school districts to implement. This is connected to a larger vision of every child graduating college and career ready. Academic standards work together with other critical principles and efforts to educate every child to graduate college and career ready. Here, the vision and set of Guiding Principles form the foundation for building a supportive process for teaching and learning rigorous and relevant content. The following sections articulate this integrated approach to increasing student success in Wisconsin schools and communities.

Relating the Academic Standards to All Students

Grade-level standards should allow ALL students to engage, access, and be assessed in ways that fit their strengths, needs, and interests. This applies to the achievement of students with IEPs (individualized education plans), bilingual learners, English learners, and gifted and talented pupils, consistent with all other students. Academic standards serve as the foundation for individualized programming decisions for all students.

Academic standards serve as a valuable basis for establishing concrete, meaningful goals as part of each student's developmental progress and demonstration of proficiency. Students with IEPs must be provided specially designed instruction that meets their individual needs. It is expected that each individual student with an IEP will require unique services and supports matched to their strengths and needs in order to close achievement gaps in grade-level standards. Alternate standards are only available for students with the most significant cognitive disabilities.

Gifted and talented students may achieve well beyond the academic standards and move into advanced grade levels or into advanced coursework.

Our Vision: Every Child a Graduate, College and Career Ready

We are committed to ensuring every child graduates from high school academically prepared and socially and emotionally competent. A successful Wisconsin student is proficient in academic content and can apply their knowledge through skills such as critical thinking, communication, collaboration, and creativity. The successful student will also possess critical habits such as

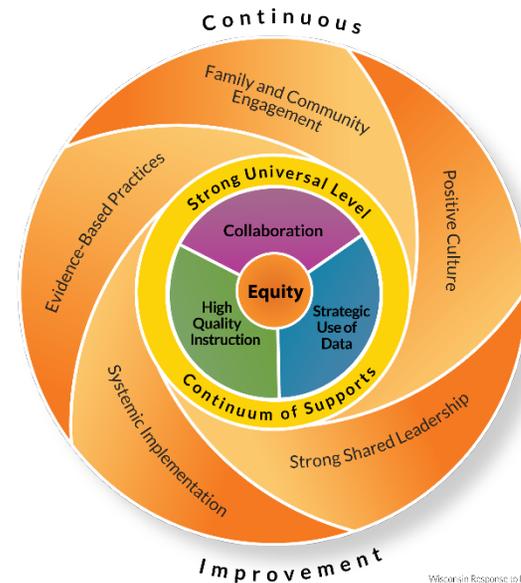
perseverance, responsibility, adaptability, and leadership. This vision for every child as a college and career ready graduate guides our beliefs and approaches to education in Wisconsin.

Guided by Principles

All educational initiatives are guided and impacted by important and often unstated attitudes or principles for teaching and learning. [The Guiding Principles for Teaching and Learning \(2011\)](#) emerge from research and provide the touchstone for practices that truly affect the vision of [Every Child a Graduate Prepared for College and Career](#). When made transparent, these principles inform what happens in the classroom, direct the implementation and evaluation of programs, and most importantly, remind us of our own beliefs and expectations for students.

Ensuring a Process for Student Success

For Wisconsin schools and districts, implementing the [Framework for Equitable Multi-Level Systems of Supports \(2017\)](#) means providing equitable services, practices, and resources to every learner based upon responsiveness to effective instruction and intervention. In this system, high-quality instruction, strategic use of data, and collaboration interact within a continuum of supports to facilitate learner success. Schools provide varying types of supports with differing levels of intensity to proactively and responsibly adjust to the needs of the whole child. These include the knowledge, skills and habits learners need for success beyond high school, including developmental, academic, behavioral, social, and emotional skills.



Connecting to Content: Wisconsin Academic Standards

Within this vision for increased student success, rigorous, internationally benchmarked academic standards provide the content for high-quality curriculum and instruction and for a strategic assessment system aligned to those standards. With the adoption of the standards, Wisconsin has the tools to design curriculum, instruction, and assessments to maximize student learning. The standards articulate what we teach so that educators can focus on how instruction can best meet the needs of each student. When implemented within an equitable multi-level system of support, the standards can help to ensure that every child will graduate college and career ready.

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Section II

Wisconsin Standards for Mathematics

What is Mathematics Education in Wisconsin?

Wisconsin's Vision for Mathematics

The Wisconsin vision for Mathematics is shaped by Wisconsin practitioners, experts, and the business community, and is informed by work at the national level and in other states. The overarching goal of Wisconsin's vision for Mathematics is for students to see themselves as confident doers and learners of mathematics, supporting the department's vision to be college and career ready.

1. Wisconsin's students will develop deep mathematics understanding so that they may experience joy and confidence in themselves as mathematicians.
2. Wisconsin's students will develop as mathematicians through both mathematical practices and content.
3. Wisconsin's students will be flexible users of mathematics as they use mathematics to understand the world and also question and critique the world using mathematical justifications.
4. Wisconsin's students will have expanded professional opportunities in a wide variety of careers.

Wisconsin's Guiding Principles for Teaching and Learning (Wisconsin Department of Public Instruction, 2011) provide important guidance for approaching the vision of mathematics. Each of the six guiding principles has implications for equity, pedagogy, instruction and assessment. Mathematics educators should consider how teaching and learning systems and structures are in service of students with respect to each of the principles.

Every student has the right to learn significant mathematics.

Mathematical proficiency is essential for every student in Wisconsin. Students need to be able to formulate, represent, and solve problems; explain and justify solutions and solution paths; and see mathematics as sensible, useful, and worthwhile. In order to achieve this vision, all students must have access to challenging, rigorous, and meaningful mathematics. Schools and classrooms need to be organized to convey the message that all students can learn mathematics and should be expected to achieve.

Mathematics instruction should be rigorous and relevant.

Teachers focus on engaging students in using mathematical reasoning, making mathematical connections, and modeling and representing mathematical ideas in a variety of ways. The mathematics curriculum needs to integrate and sequence important mathematical ideas so that mathematics makes sense. Teachers use rich tasks to engage students in the development of conceptual understanding and procedural skills. An emphasis on connections within mathematics helps students see

mathematics as a coherent and integrated whole rather than as a set of isolated and disconnected skills and procedures. Through mathematical applications, students recognize the usefulness of mathematics and appreciate the need to study and understand mathematical skills and concepts.

Purposeful assessment drives mathematics instruction and affects learning.

Teachers measure mathematical proficiency by using a variety of purposeful assessments before, during, and after instruction. Rich assessment tasks ask students to demonstrate their understanding by representing mathematical situations, solving problems as developed in the classroom, and justifying their solutions. Valuable assessments provide both students and teachers with the opportunity to reflect on students' mathematical communication, precision, and reasoning. Teachers use resulting data to adapt their instruction and the learning environment so that all students will understand new mathematics concepts and content.

Learning mathematics is a collaborative responsibility.

Collaborative structures, within the mathematics classroom as well as in the school community, support the teaching and learning of mathematics. Students develop mathematical habits of mind through purposeful interactions in the classroom. Teachers co-create contexts, conditions, and assessment strategies for an interdependent learning environment. Opportunities for students to communicate the solutions, solution paths, and justifications are present in mathematics lessons.

Students bring strengths and experiences to mathematics learning.

Students bring informal experiences of mathematics from their home and community to the mathematics classroom. They may enter classrooms with varying levels of mathematical misconceptions and confidence in their ability to do mathematics. Schools and teachers must build upon students' prior knowledge and intuitive understanding of mathematical ideas in order to connect the formal study of mathematics to students' ongoing experiences. Teachers need to continually identify students' strengths and weaknesses as a basis to develop tasks and experiences that will capitalize on student strengths and address weaknesses and misconceptions.

Responsive environments engage mathematics learners.

Teachers utilize strategies that create effective mathematics environments. These environments use high quality mathematics curriculum and instruction in response to the understanding that not all students learn at the same pace or in the same way. Student engagement, perseverance, and learning are increased when teachers respond to students' interests, learning profiles,

and readiness. The Standards for Mathematical Practice are evident in a responsive environment.

Wisconsin Standards for Mathematics (2021) provide a description, or portrait, of students who have met the standards in mathematics both in content and practices.

These standards articulate end-of-grade level expectations. All students deserve access to grade-level instructional materials. Some students - including students who receive special education services through an Individualized Education Program (IEP), students who receive reasonable accommodations through a 504 plan, English language learners, and emerging bi-literate students - may benefit from additional supports. Students with gifts and talents often require further academic challenges. Some barriers to learning and engagement can be minimized through Universal Design for Learning (UDL). In addition, learning can be personalized through collaboration between educators, school staff, families, and students.

At the elementary level, mathematics content and concepts can be integrated throughout the curriculum and should be connected to student's lives. Teachers can effectively use mathematics concepts in instruction to develop foundational skills and also can create a connection to secondary mathematics options. At the middle and high school levels, all students should have access to mathematics, including those who wish to pursue advanced courses.

Wisconsin's Approach to Academic Standards for Mathematics

The *Wisconsin Standards for Mathematics (2021)* are built on the foundation of existing standards (Council of Chief State School Officers, 2010) and incorporate shifts that reflect new research and broader expectations of Mathematics. Three of the five shifts are from the *Wisconsin Standards for Mathematics (2010)*, but have been expanded upon to emphasize advancing educational equity in mathematics. Two of the five shifts are new and unique to Wisconsin. There are five important shifts from previous standards (2010) to these revised *Wisconsin Standards for Mathematics (2021)*. Identifying the key shifts builds understanding of how these standards differ from previous standards. The shifts also serve to guide educators in identifying what is necessary in standards-aligned instruction and assessment.

Shift #1: Learning mathematics emphasizes recognizing, valuing, and fostering mathematical identities and agency in all students.

The *Wisconsin Standards for Mathematics (2021)* expects opportunities for inclusion of broader ways to think and do mathematics. This shift supports recognizing and valuing the mathematical ways of thinking students bring with them to school mathematics from their culture, their families or previous grade level. By leveraging multiple mathematical competencies,

drawing on multiple resources of knowledge, and going deep into the mathematical concepts students develop stronger mathematical understanding (Aguirre 2013, 43).

Shift #2: All students are flexible users of mathematics who see how mathematics can be used to understand their world and the world around them.

The *Wisconsin Standards for Mathematics* (2021) call for empowering students to be the thinkers and doers of mathematics. The standards are calling for an intentional pairing of the Standards for Mathematical Practice and the Standards for Mathematical Content that allow for students to gain a lifelong appreciation of mathematics and how mathematics is used to critique and understand the world.

Mathematical modeling is a powerful way to bring this shift to life in students' mathematical journeys. Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena (GAIMME 2016, 8).

Shift #3: All students engage in mathematics that is focused on developing deep understanding of and connections among mathematical concepts in order to gain strong foundations to move their mathematics learning forward.

The *Wisconsin Standards for Mathematics* (2021) continue to call for a focus in mathematics content at each grade level. The standards significantly narrow and deepen the way time and energy are spent. This means focusing deeply on the major work of each grade as follows:

- In grades K–2: Concepts, skills, and problem solving related to addition and subtraction
- In grades 3–5: Concepts, skills, and problem solving related to multiplication and division of whole numbers and fractions
- In grade 6: Ratios and proportional relationships, and early algebraic expressions and equations
- In grade 7: Ratios and proportional relationships, and arithmetic of rational numbers
- In grade 8: Linear algebra and linear functions

The *Wisconsin Standards for Mathematics* (2021) focus deeply on the major work of each grade. This focus will help students gain strong foundations, including a solid understanding of concepts to support their mathematical experiences. Students develop a strong foundational knowledge and deep conceptual understanding and are able to transfer mathematical skills and understanding across concepts and grades. (NMAP 2008, 15-20)

Shift #4: All students engage in coherent mathematics that connects concepts and mathematical thinking within and

across domains and grades.

Mathematics is not a list of disconnected topics, tricks, or mnemonics; it is a coherent body of knowledge made up of interconnected concepts. Therefore, the standards are designed around coherent progressions from grade to grade. Learning is carefully connected across grades so that students can build new understanding onto foundations built in previous years. Coherence is also built into the standards in how they reinforce a major topic in a grade by utilizing supporting, complementary topics.

Shift #5: All students engage in rigorous mathematics focused with equal intensity on developing conceptual understanding, procedural flexibility and efficiency and application to authentic contexts.

Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, flexible and efficient procedural skills, and application.

Conceptual understanding: The standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to access concepts from a number of perspectives in order to see math as more than a set of mnemonics or discrete procedures.

Flexible and efficient procedural skills: The standards call for efficiency and accuracy in calculation. Flexible and efficient procedural skills build from an initial exploration and discussion of number concepts to using informal reasoning strategies and the properties of operations to develop general methods for solving problems (NCTM 2014). Students must have a command of core functions, such as single-digit multiplication, in order to have access to more complex concepts and procedures.

Application: The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students having a solid conceptual understanding and procedural flexibility and efficiency.

The *Wisconsin Standards for Mathematics* (2021) may be taught and integrated through a variety of classes and experiences. Each district, school, and program area should determine the means by which students meet these standards. Through the collaboration of multiple stakeholders, these foundational standards will set the stage for high-quality, successful, contemporary mathematics courses and programs throughout Wisconsin's PK-12 systems.

More information regarding the Wisconsin's Key Shifts in Mathematics and how they impact instruction and instructional materials can be found in Appendix 1.

Content Standards Structure

Wisconsin Standards for Mathematics have the following design features:

- **Domains** - Larger groups of related standards. Standards from different domains may sometimes be closely related. In high school standards there is an additional grouping of domains called **Conceptual Categories**.
- **Clusters** - Groups of related standards. Notice that the cluster statements appear at the far left of each standard table to visually highlight an emphasis on the cluster statement. Individual standards stem from cluster statements and provide the details of what students should understand and be able to do. Standards from different clusters may sometimes be closely related, because mathematics is a connected subject.
- **Standards** - Define what students should understand and be able to do.
 - **New Standard Numbers** - When standard numbers were changed from those used in 2010 due to the addition of a new standard within an existing cluster, the new standard code appears (e.g., M.K.CC.C.7) as well as the previous code. (e.g., [WI.2010.K.CC.C.6] for reference).
 - **(M) Mathematical Modeling** - Mathematical modeling is best interpreted, not as a collection of isolated topics, but rather in relation to other standards. To support these relationships, content standards that may be particularly valuable in high school have been indicated with an (M) symbol. The (M) symbol may appear following the conceptual category, domain, cluster or standard. If the (M) appears at a larger group level, it should be understood that the designation applies to all standards in that group.

Domain

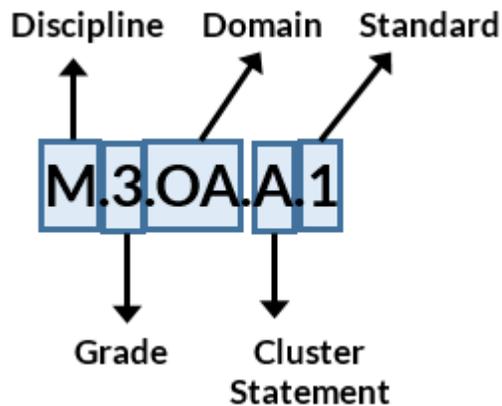
Number and Operations in Base Ten (3.NBT)

Cluster Statement	Notation	Standard
A. Use place value understanding and properties of operations to perform multi-digit arithmetic, using a variety of strategies.	M.3.NBT.A.1	Use place value understanding to generate estimates for problems in real-world situations, with whole numbers within 1,000, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates (e.g., Is my estimate too low or too high? What degree of precision do I need for this situation?).
	M.3.NBT.A.2	Flexibly and efficiently add and subtract within 1,000 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
	M.3.NBT.A.3	Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9×80 , 5×60) using strategies based on place value and properties of operations.

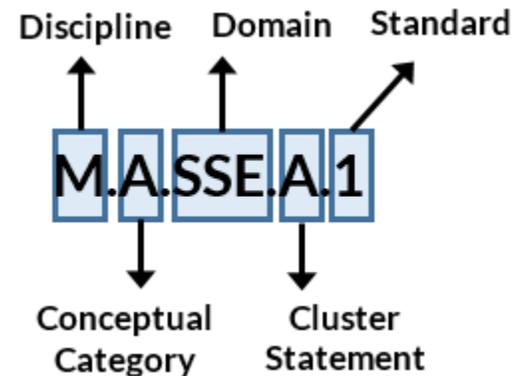
Standard

Cluster

K-8 Standards Coding



High School Standards Coding



Focus and Organization of the K-12 Domains and Conceptual Categories

The mathematics content of the *WI Standards for Mathematics* builds across grades and provides important underpinnings for the mathematics to be learned at subsequent levels. The coherence of the *WI Standards for Mathematics* lies in those connections, both within and across grade levels and topics. The graphic below illustrates the domains and conceptual categories of the *WI*

At the early elementary grades, the focus is largely on the areas of number and operations in base ten and algebraic thinking. This expands to a focus on fractions later in elementary school. The K-5 mathematics content provides the groundwork for the study of ratios, proportional reasoning, the number system, expressions and equations, and functions at the middle school level. By providing a focused mathematics experience in elementary and middle school, a strong foundation is developed for the content to be learned at the high school level. “Mathematical modeling should be taught at every stage of a student’s mathematical education” (GAIMME 2016, 7). “Mathematics is important for its own sake, but mostly.... mathematics is important in dealing with the rest of the world. Certainly, mathematics will help students as they move on through school and into the world of work. but it can and should help them in their daily lives and as informed citizens” (GAIMME 2016, 7).

Appendices/Additional Resources follow both the Standards for Mathematical Practice and the Content Standards for grades K-12:

- **Wisconsin’s Key Shifts in Mathematics** - Outlines the five important shifts from previous standards to the *Wisconsin Standards for Mathematics* (2021) and supports understanding in how these standards differ. Three of the five shifts are from the 2010 standards but have been expanded upon to emphasize current research and advancing educational equity in mathematics. Two of the five shifts are new and unique to Wisconsin.
- **Glossary** - Supports the understanding of all grade level standards.
- **Tables** - Provides problem situation types and examples as well as the properties of operations, equality, and inequality. Notice that tables 1, 2A, and 2B have been updated in *Wisconsin Standards for Mathematics* (2021).
- **Mathematical Modeling** - Identifies common elements across K-12 modeling and provides a continuum of learning.
- **High School** - Supports how standards can be used to build high school courses and sequences of courses.

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Section III

Discipline: Mathematics

Standards

These revised state standards (2021), demonstrate the belief that every student has the ability to develop deep mathematical understanding as a confident and capable learner. To achieve this and develop strong mathematical identities in the process, all students need access to grade level standards. It is important that users of this document keep the five Wisconsin Key Shifts at the forefront of their minds while interacting with both the Standards for Mathematical Practice and the Standards for Mathematical Content, K-12.

Standards for Mathematical Practice

The Standards for Mathematical Practice are central to the teaching and learning of mathematics. These practices describe the behaviors and habits of mind that are exhibited by students who are mathematically proficient. Mathematical understanding is the intersection of these practices and mathematics content. It is critical that the Standards for Mathematical Practice are embedded in daily mathematics instruction.

The Standards for Mathematical Practice include:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments, and appreciate¹ and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The *Wisconsin Standards for Mathematics* (2021) includes K-12 versions of the Standards for Mathematical Practice as well as grade band versions of these same standards. The K-12 Standards for Mathematical Practice illustrate that these habits of mind

are consistent at all levels of mathematics and develop over time. The grade band versions of the Standards for Mathematical Practice aim to provide examples of how these practices might come to life within grade level content, providing additional clarity. The writing team benefited from earlier efforts to make the practice standards grade-band specific. For example, the work of Massachusetts provided a starting point.²

Standards for Mathematical Content

The Standards for Mathematical Content describe the sequence of important mathematics content that students learn. They are a combination of procedures and understandings. These content standards are organized around domains and clusters which are specified by grade level, kindergarten through grade 8, and by conceptual category at high school. The domains at all levels are based on research-based learning progressions detailing what is known about students' mathematical knowledge, skill and understanding. The progressions build from grade to grade and topic to topic, providing K-12 focus and coherence, but these standards do not dictate curriculum or teaching methods. For example, just because topic A appears before topic B in the standards for a given grade, it does not necessarily mean that topic A must be taught before topic B. Other important cross-grade themes that should be noted and investigated are concepts such as the role of units and unitizing, the properties of operations across arithmetic and algebra, operations and the problems they solve, transformational geometry, reasoning and sense-making, and modeling of and with mathematics.

The introductions at each K-8 grade level specify two to four key areas that are identified as critical areas of instruction and discuss the use of mathematical modeling in supporting students in becoming flexible users of mathematics at that grade level. At the high school level, the introductions describe the focus for each conceptual category, as well as the connections to other categories and domains.

Learning mathematics with understanding is a focus of the *Wisconsin Standards for Mathematics* (2021). Many of the Standards for Mathematical Content begin with the verb “understand” and are crucial for mathematical proficiency. It is generally agreed that students understand a concept in mathematics if they can use mathematical reasoning with a variety of representations and connections to explain the concept to someone else or apply the concept to another situation. This is how ‘understand’ should be interpreted when implementing the *Wisconsin Standards for Mathematics* (2021).

While the *Standards for Mathematical Practice* should be addressed with all of the Standards for Mathematical Content, the content standards that begin with the verb “understand” are a natural intersection between the two.

Understanding Mathematics

Wisconsin Standards for Mathematics (2021) define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood

it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as $(a + b)(x + y)$ and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding $(a + b + c)(x + y)$. Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness.

Wisconsin Standards for Mathematics (2021) set grade-specific standards but do not define the intervention methods or materials necessary to support students who are well below or well above grade-level expectations. It is also beyond the scope of these standards to define the full range of supports appropriate for bilingual learners, English language learners, and for students with special needs. At the same time, all students must have the opportunity to learn and meet the same high standards if they are to access the knowledge and skills necessary in their post-school lives. *Wisconsin Standards for Mathematics (2021)* should be read as allowing for the widest possible range of students to participate fully from the outset, along with appropriate accommodations to ensure maximum participation of students with special education needs. For example, for students with disabilities reading should allow for use of Braille, screen reader technology, or other assistive devices, while writing should include the use of a scribe, computer, or speech-to-text technology. In a similar vein, speaking and listening should be interpreted broadly to include sign language. No set of grade-specific standards can fully reflect the great variety in abilities, needs, learning rates, and achievement levels of students in any given classroom. However, these standards do provide clear signposts along the way to the goal of college and career readiness for all students.

Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

K-12 Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Math Practice 2: Reason abstractly and quantitatively.

K-12 Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Math Practice 3: Construct viable arguments and appreciate and critique the reasoning of others. (Gutierrez 2017)

K-12 Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine

domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Math Practice 4: Model with mathematics.

K-12 Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Math Practice 5: Use appropriate tools strategically.

K-12 Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Math Practice 6: Attend to precision.

K-12 Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the

elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Math Practice 7: Look for and make use of structure.

K-12 Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y .

Math Practice 8: Look for and express regularity in repeated reasoning.

K-12 Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

K-5 Elementary Standards

This section includes K-5 specific Standards for Mathematical Practice as well as grade level introductions, overviews, and Standards for Mathematical Content for each grade, kindergarten through fifth.

K-5 Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

K-5 Mathematically proficient elementary students explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, instead of hunting for “key words” in a word problem, students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets. If students are not at first making sense of a problem or seeing a way to begin, they ask questions about what is happening in the problem that will help them get started. As they work, they continually ask themselves, “Does this make sense?” When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, they might replace multi-digit numbers in a word problem with single-digit numbers to better appreciate the quantities in the problem and how they relate.

Once students have a solution, they often check their answers to problems using a different approach. Mathematically proficient students consider different solution pathways, both their own and those of other students, in order to identify and analyze connections among approaches. They can explain connections among physical models, pictures, diagrams, equations, verbal descriptions, tables, and graphs.

Math Practice 2: Reason abstractly and quantitatively.

K-5 Mathematically proficient elementary students make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using visual representations or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent.

Mathematically proficient students know and flexibly use different properties of operations, numerical relationships, numbers, and geometric objects. They can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, if a student chooses to evaluate the expression 13×25 mentally, the student might think of a context to help produce a strategy—for example, by thinking “Thirteen groups of 25 is like having 13 quarters.” This prompts a strategy of thinking “I know that 10 quarters is \$2.50 and 3 quarters is \$0.75. \$2.50 and \$0.75 is \$3.25.” In this example the student uses a context to think through a strategy for solving the problem, using their knowledge of money and of decomposing one factor based on place value ($13 = 10 + 3$). The student then uses the context to identify the solution to the original problem.

Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context. Consider the problem: A teacher wants to bring 10 pumpkins to school to decorate the classroom. Some are big pumpkins and some are small pumpkins. How many of each size pumpkin might the teacher bring to school? When students create the number sentence $4+6=10$, they have decontextualized the problem and expressed it with numbers and symbols. When they can explain that the number sentence means, “4 big pumpkins plus 6 small pumpkins equals 10 pumpkins,” they demonstrate their ability to recontextualize the numbers and equation back to the word problem.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others. (Gutierrez 2017)

K-5 Mathematically proficient elementary students construct verbal and written mathematical arguments that explain the reasoning underlying a strategy, solution, or conjecture. Arguments might use concrete referents such as objects, drawings, diagrams, and actions. Arguments might also rely on definitions, previously established results, properties, or structures. For example, a student might argue that $\frac{1}{5} > \frac{1}{9}$ on the basis that one of 5 equal parts of a whole is larger than one of 9 equal parts of that whole, because with more equal parts, the size of each part must be smaller. Another example is reasoning that two different shapes have equal area because it has already been demonstrated that they are each half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

Mathematically proficient students present their arguments in the form of representations, actions on those representations, explanations in words (oral or written), or a combination of these three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, primary grade students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. Intermediate grade students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals.

Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments both orally and in writing, compare them to others, and reconsider their own arguments in response to the critiques of others.

Math Practice 4: Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life. Mathematically

proficient elementary students can identify the mathematical elements of a situation and generate questions that can be addressed using mathematics (e.g., noticing and wondering). They can create and interpret mathematical models that show those elements and relationships among them. Although elementary students often use manipulatives and drawings when learning to understand calculations, this modeling mathematical practice refers specifically to the modeling of contextual situations.

Mathematical models might be represented in one or more of the following ways: numbers and symbols; geometric figures, pictures, or physical objects used to abstract the mathematical elements of the situation; a mathematical diagram such as a number line, table, or graph. In order to help young students mathematize their world, they can be asked to consider how the classroom's set of blocks should be shared throughout recess time. Students would then need to make assumptions about how many blocks each student should have as well as how many minutes each student should have the blocks. Students could then draw a plan to represent their mathematical model. Students could be asked to refine their model by posing the question, "What if one of our friends will not be at recess?"

Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multi-step problems such as those involving more than one unknown quantity.

Mathematically proficient students use and interpret models to analyze relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

Math Practice 5: Use appropriate tools strategically.

K-5 Mathematically proficient elementary students strategically consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (e.g., manipulatives, pencil and paper, rulers); conceptual tools (e.g., properties of operations, algorithms); drawings or diagrams (e.g., number lines, tally marks, tape diagrams, arrays, tables, graphs) and available technologies (e.g., calculators, online apps).

Mathematically proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations. Students choose tools that are relevant and useful to the problem at hand. For example, when determining how to measure length, students may compare the benefits of using non-standard units of measure (e.g., their own hands, paperclips) versus standard units and tools (e.g., an inch or centimeter ruler). As another example, when presented with 1002-3 or 101-98, students subtract strategically, which may involve reasoning, counting, or decomposing rather than using a written algorithm.

Math Practice 6: Attend to precision.

K-5 Mathematically proficient elementary students use precise language to communicate orally and in written form. They come to appreciate, understand, and use mathematical vocabulary not in isolation, but in the context of doing mathematical thinking and problem solving. They may start by using everyday language to express their mathematical ideas, and gradually select words with greater clarity and specificity. For example, they may initially use the word “triangle” to refer only to equilateral triangles resting on their bases, but come to understand and use a more precise definition of a triangle as a closed figure with three straight sides. As another example, they may initially explain a solution by saying, “it works” without explaining what “it” means but later clarify their explanation with specific details.

In using mathematical representations, students provide appropriate labels to precisely communicate the meaning of their representations (e.g., charts, graphs, and drawings). When making mathematical arguments about a solution, strategy, or conjecture (MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations.

Students use mathematical symbols correctly and can describe the meaning of the symbols they use. For example, they use the equal sign consistently and appropriately. They state the meaning of the symbols they choose in relation to the problem at hand.

Students use tools and strategies (e.g., measuring tools, estimation) effectively, to maintain a level of precision that is appropriate to the situation. They specify units of measure where needed.

Perseverance and attention to detail are mathematical habits of mind; mathematically proficient students check for reasonableness and accuracy by solving a problem a second way, analyzing errors and learning from them.

Math Practice 7: Look for and make use of structure.

K-5 Mathematically proficient elementary students use structures such as place value, the properties of operations, and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). When students use an algorithm to solve $53 \cdot 17$ in order to fully understand how to decompose the tens and ones, they must understand that 53 can be seen as 4 tens and 13 ones, not just 5 tens and 3 ones.

When younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16×9 , they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10 + 6) \times 9 = (10 \times 9) + (6 \times 9)$. To determine the volume of a $3 \times 4 \times 5$ rectangular prism, students might see the structure of the prism as five layers of 3×4 arrays of cubes.

Students in elementary grades look for and make use of structure when they view expressions as objects to observe and interpret. For example, students might observe that $120 - 41$ must be one less than $120 - 40$ because “if you subtract one more, the result will be one less” (MP.8). Students can interpret the expression $5 \times 3 + 6 \times 3$ as “five groups of three and six more groups of three” or notice there are a total of 11 groups of 3.

A word problem that involves distributing 29 marbles among 4 vases could lead (MP.4) to an equation model $(29 - 1) \div 4 = 7$, where the expression on the left-hand side not only has the value 7 but also suggests, based on its structure, a process of discarding 1 marble and dividing the rest of the marbles equally into 4 groups of 7.

Math Practice 8: Look for and express regularity in repeated reasoning.

K-5 Mathematically proficient elementary students look for and identify regularities as they solve multiple related problems. Students make and test conjectures, reason about and express these regularities as generalizations about structures and relationships, and then use those generalizations to solve problems (MP.7).

For example, younger students might notice that when tossing two-color counters to find combinations of a given number, over time students will notice a pattern (commutative property of addition). For example, when tossing six 2-sided counters, they may get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow.

In the elementary grades students can recognize and use patterns to help them become flexible with addition. For example, given the number string below, students may recognize they can take one away from the 5 and add it to the first number to make a multiple of ten. They also may notice a pattern related to the first digit increasing by 10, therefore the answer increases by 10.

$$9+5 \quad 19+5 \quad 29+5 \quad 39+5$$

When drawing and representing fractions, students might notice a consistent relationship between the numerator and denominator of fractions that equal one half (e.g., that the numerator is half the denominator and the denominator is two times the numerator). They can generalize from these repeated examples that all fractions equal to one half show this relationship.

As students practice articulating their observations both verbally and in writing, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).

Introduction: Kindergarten

In Kindergarten, instructional time should focus on two critical areas: representing and comparing whole numbers, initially with sets of objects, and describing shapes and space. More learning time in Kindergarten should be devoted to number than to other topics. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade.²

Students use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set; counting out a given number of objects; comparing sets or numerals; and modeling simple joining and separating situations with sets of objects, or eventually with equations such as $5 + 2 = 7$ and $7 - 2 = 5$. (Kindergarten students should see addition and subtraction equations, and student writing of equations in kindergarten is encouraged, but it is not required.) Students choose, combine, and apply effective strategies for answering quantitative questions, including subitizing or quickly recognizing the cardinalities of small sets of objects, counting and producing sets of given sizes, counting the number of objects in combined sets, or counting the number of objects that remain in a set after some are taken away.

Students describe their physical world using geometric ideas (e.g., shape, orientation, spatial relations) and vocabulary. They identify, name, and describe basic two-dimensional shapes, such as squares, triangles, circles, rectangles, and hexagons, presented in a variety of ways (e.g., with different sizes and orientations), as well as three-dimensional shapes such as cubes, cones, cylinders, and spheres. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the primary level, the big idea within mathematical modeling is showing young students how to frame, explore, and answer relevant real-world questions using mathematical ideas. A class focuses on a central open question, possibly motivated by student interest, but that then is identified, selected, or articulated by the teacher. Students gather and record data to support answering the question by using

mathematics. Graphing may be done as a whole class until students are ready to try it independently. Once a visual is created, the class will decide what information or conclusions can be drawn from the representations of the information. The class might even decide they want to collect more information about this or another related idea after looking at the information.³

Grade K Overview

Counting and Cardinality

- Know number names and the count sequence.
- Tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11-19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in each category.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Kindergarten Content Standards

Counting and Cardinality (K.CC)

Cluster Statement	Notation	Standard
A. Know number names and the count sequence.	M.K.CC.A.1	Count to 100 by ones and by tens.
	M.K.CC.A.2	Count forward beginning from a given number within the known sequence (instead of having to begin at 1).
	M.K.CC.A.3	Write numbers from 0 to 20. Represent a number of objects with a written numeral 0-20 (with 0 representing a count of no objects).
B. Tell the number of objects.	M.K.CC.B.4	<p>Understand the relationship between numbers and quantities; connect counting to cardinality.</p> <p>a. When counting objects, say the number names in the standard order, pairing each object with one and only one number name and each number name with one and only one object (one to one correspondence).</p> <p>b. Understand that the last number name said tells the number of objects counted (cardinality). The number of objects is the same regardless of their arrangement or the order in which they were counted (number conservation).</p> <p>c. Understand that each successive number name refers to a quantity that is one large (hierarchical inclusion).</p>
	M.K.CC.B.5	Quickly recognize and name the quantity of up to 5 objects briefly shown in structured or unstructured arrangements without counting (perceptual subitizing).

	M.K.CC.B.6 [WI.2010. K.CC.B.5]	Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1-20, count out that many objects.
C. Compare numbers.	M.K.CC.C.7 [WI.2010. K.CC.C.6]	Identify whether the number of objects (up to 10) in one group is greater than, less than, or equal to the number of objects in another group, e.g., by using matching and counting strategies.
	M.K.CC.C.8 [WI.2010. K.CC.C.7]	Compare two numbers between 1 and 10 presented as written numerals.

Operations and Algebraic Thinking (K.OA)

Cluster Statement	Notation	Standard
A. Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.	M.K.OA.A.1	Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, or numbers. Drawings need not show details, but should show the mathematics in the problem.
	M.K.OA.A.2	Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem. <i>See Appendix, Table 1 for specific problem situations and category information.</i>
	M.K.OA.A.3	Compose and decompose quantities within 10 <ul style="list-style-type: none"> a. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition with drawings or numbers. b. Quickly name the quantity of objects briefly shown in structured arrangements anchored to 5 (e.g., fingers, ten frames, math rack/rekenrek) with totals up to 10 without counting by recognizing the arrangement or seeing the quantity in subgroups that are combined (conceptual subitizing).
	M.K.OA.A.4	For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or numbers.
	M.K.OA.A.5	Flexibly and efficiently add and subtract within 5 using mental images and composing/decomposing numbers up to 5.

Number and Operations in Base Ten (K.NBT)

Cluster Statement	Notation	Standard
A. Work with numbers 11-19 to gain foundations for place value.	M.K.NBT.A.1	Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or numbers; understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.

Measurement and Data (K.MD)

Cluster Statement	Notation	Standard
A. Describe and compare measurable attributes.	M.K.MD.A.1	Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
	M.K.MD.A.2	Directly compare two objects with a measurable attribute in common, to see which object has "more of" / "less of" the attribute, and describe the difference. <i>For example, directly compare the heights of two children and describe one child as taller/shorter.</i>
B. Classify objects and count the number of objects in each category.	M.K.MD.B.3	Classify objects into given categories; count the numbers of objects in each category and sort the categories by count. Limit category counts to be less than or equal to 10.

Geometry (K.G)

Cluster Statement	Notation	Standard
A. Identify and describe shapes (squares, circles, triangles, rectangles, hexagons, cubes, cones, cylinders, and spheres).	M.K.G.A.1	Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to.
	M.K.G.A.2	Correctly name shapes regardless of their orientations or overall size.
	M.K.G.A.3	Identify shapes as two-dimensional (lying in a plane, "flat") or three-dimensional ("solid").
B. Analyze, compare, create, and compose shapes.	M.K.G.B.4	Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts (e.g., number of sides and vertices/"corners") and other attributes (e.g., having sides of equal length).
	M.K.G.B.5	Model shapes in the world by building shapes from components (e.g., sticks and clay balls) and drawing shapes.
	M.K.G.B.6	Compose simple shapes to form larger shapes. <i>For example, "Can you join these two triangles with full sides touching to make a rectangle?"</i>

Introduction: Grade 1

In Grade 1, instructional time should focus on four critical areas: developing understanding of addition, subtraction, and strategies for addition and subtraction within 20; developing understanding of whole number relationships and place value, including grouping in tens and ones; developing understanding of linear measurement and measuring lengths as iterating length units; and reasoning about attributes of, and composing and decomposing geometric shapes. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the *Standards for Mathematical Practice*. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students develop strategies for adding and subtracting whole numbers based on their prior work with small numbers. They use a variety of models, including discrete objects and length-based models (e.g., cubes connected to form lengths), to model add-to, take-from, put-together, take-apart, and compare situations to develop meaning for the operations of addition and subtraction, and to develop strategies to solve arithmetic problems with these operations. Students understand connections between counting and addition and subtraction (e.g., adding two is the same as counting on two). They use properties of addition to add whole numbers and to create and use increasingly sophisticated strategies based on these properties (e.g., “making tens”) to solve addition and subtraction problems within 20. By comparing a variety of solution strategies, children build their understanding of the relationship between addition and subtraction.

Students develop, discuss, and use efficient, accurate, and generalizable methods to add within 100 and subtract multiples of 10. They compare whole numbers (at least to 100) to develop understanding of and solve problems involving their relative sizes. They think of whole numbers between 10 and 100 in terms of tens and ones (especially recognizing the numbers 11 to 19 as composed of a ten and some ones). Through activities that build number sense, they understand the order of the counting numbers and their relative magnitudes.

Students develop an understanding of the meaning and processes of measurement, including underlying concepts such as iterating (the mental activity of building up the length of an object with equal-sized units) and gain experiences that allow them to intuit the transitivity principle for indirect measurement. (Students should apply the principle of transitivity of measurement to make indirect comparisons, but they need not use this technical term. See the glossary for the definition of transitivity principle for indirect measurement.)

Students compose and decompose plane or solid figures (e.g., put two triangles together to make a quadrilateral) and build understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine shapes, they recognize them from different perspectives and orientations, describe their geometric attributes, and determine how they are alike and different, to develop the background for measurement and for initial understandings of properties such as congruence and symmetry.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the primary level, the big idea within mathematical modeling is showing young students how to frame, explore, and answer relevant real-world questions using mathematical ideas. A class focuses on a central open question, possibly motivated by student interest, but that then is identified, selected, or articulated by the teacher. Students gather and record data to support answering the question by using mathematics. Graphing may be done as a whole class until students are ready to try it independently. Once a visual is created, the class will decide what information or conclusions can be drawn from the representations of the information. The class might even decide they want to collect more information about this or another related idea after looking at the information (GAIMME 2016, 32).

Grade 1 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Understand and apply properties of operations and the relationship between addition and subtraction.
- Add and subtract within 20.
- Work with addition and subtraction equations.

Number and Operations in Base Ten

- Extend the counting sequence.
- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure lengths indirectly and by iterating length units.
- Tell and write time.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 1 Content Standards

Operations and Algebraic Thinking (1.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving addition and subtraction.	M.1.OA.A.1	Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. <i>See Appendix, Table 1 for specific problem situations and category information.</i>
	M.1.OA.A.2	Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.
B. Understand and apply properties of operations and the relationship between	M.1.OA.B.3	Apply properties of operations as strategies to add and subtract. <i>Examples: If $8 + 3 = 11$ is known, then $3 + 8 = 11$ is also known. (Informal use of the commutative property of addition.) To add $2 + 6 + 4$, the second two numbers can be added to make a ten, so $2 + 6 + 4 = 2 + 10 = 12$. (Informal use of the associative property of addition.)</i>

addition and subtraction.	M.1.OA.B.4	Understand subtraction as an unknown-addend problem. <i>For example, subtract $10 - 8$ by finding the number that makes 10 when added to 8.</i>
C. Add and subtract within 20.	M.1.OA.C.5	Use counting and subitizing strategies to explain addition and subtraction. a. Relate counting to addition and subtraction (e.g., by counting on 2 to add 2). b. Use conceptual subitizing in unstructured arrangements with totals up to 10 and structured arrangements anchored to 5 or 10 (e.g., 10 frames, double ten frames, math rack/rekenrek) with totals up to 20 to relate the compositions and decompositions to addition and subtraction.
	M.1.OA.C.6	Use multiple strategies to add and subtract within 20. a. Flexibly and efficiently add and subtract within 10 using strategies that may include mental images and composing/decomposing up to 10. b. Add and subtract within 20 using objects, drawings or equations. Use multiple strategies that may include counting on; making a ten (e.g., $8 + 6 = 8 + 2 + 4 = 10 + 4 = 14$); decomposing a number leading to a ten (e.g., $13 - 4 = 13 - 3 - 1 = 10 - 1 = 9$); using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).
D. Work with addition and subtraction equations.	M.1.OA.D.7	Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. <i>For example, which of the following equations are true and which are false? $6 = 6$, $7 = 8 - 1$, $5 + 2 = 2 + 5$, $4 + 1 = 5 + 2$.</i>

Number and Operations in Base Ten (1.NBT)

Cluster Statement	Notation	Standard
A. Extend the counting sequence.	M.1.NBT.A.1	Count to 120, starting at any number less than 120. In this range, read and write numerals and represent a number of objects with a written numeral.
B. Understand place value.	M.1.NBT.B.2	Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases: <ul style="list-style-type: none"> a. 10 can be thought of as a bundle of ten ones -- called a "ten". b. The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones. c. The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).
	M.1.NBT.B.3	Compare two two-digit numbers based on meanings of the tens and ones digits and describe the result of the comparison using words and symbols ($>$, $=$, and $<$).
C. Use place value understanding and properties of operations to add and subtract.	M.1.NBT.C.4	Add within 100, including adding a two-digit number and a one-digit number, and adding a two-digit number and a multiple of 10, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten.
	M.1.NBT.C.5	Given a two-digit number, mentally find 10 more or 10 less than the number, without having to count; explain the reasoning used.

	M.1.NBT.C.6	Subtract multiples of 10 in the range 10-90 from multiples of 10 in the range 10-90 (positive or zero differences), using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
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Measurement and Data (1.MD)

Cluster Statement	Notation	Standard
A. Measure lengths indirectly and by iterating length units.	M.1.MD.A.1	Order three objects by length; compare the lengths of two objects indirectly by using a third object.
	M.1.MD.A.2	Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. <i>Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.</i>
B. Tell and write time.	M.1.MD.B.3	Tell and write time in hours and half-hours using analog and digital clocks.
C. Represent and interpret data.	M.1.MD.C.4	Organize, represent, and interpret data with up to three categories; ask and answer questions about the total number of data points, how many in each category, and how many more or less are in one category than in another.

Geometry (1.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.1.G.A.1	Distinguish between defining attributes (e.g., triangles are closed and three-sided) versus non-defining attributes (e.g., color, orientation, overall size); build and draw shapes to possess defining attributes.
	M.1.G.A.2	Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. Student use of formal names such as "right rectangular prism" is not expected.
	M.1.G.A.3	Partition circles and rectangles into two and four equal shares, describe and count the shares using the words <i>halves</i> and <i>fourths</i> , and use the phrases <i>half of</i> and <i>fourth of the whole</i> . Describe the whole as being two of the shares, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

Introduction: Grade 2

In Grade 2, instructional time should focus on four critical areas: extending understanding of base-ten notation; building flexibility and efficiency with addition and subtraction; using standard units of measure; and describing and analyzing shapes. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students extend their understanding of the base-ten system. This includes ideas of counting in fives, tens, and multiples of hundreds, tens, and ones, as well as number relationships involving these units, including comparing. Students understand multi-digit numbers (up to 1000) written in base-ten notation, recognizing that the digits in each place represent amounts of thousands, hundreds, tens, or ones (e.g., 853 is 8 hundreds + 5 tens + 3 ones).

Students use their understanding of addition to develop flexibility and efficiency with addition and subtraction within 100. They solve problems within 1000 by applying their understanding of models for addition and subtraction, and they develop, discuss, and use efficient, accurate, and generalizable methods to compute sums and differences of whole numbers in base-ten notation, using their understanding of place value and the properties of operations. They select and accurately apply methods that are appropriate for the context and the numbers involved to mentally calculate sums and differences for numbers with only tens or only hundreds.

Students recognize the need for standard units of measure (centimeter and inch) and they use rulers and other measurement tools with the understanding that linear measure involves an iteration of units. They recognize that the smaller the unit, the more iterations they need to cover a given length.

Students describe and analyze shapes by examining their sides and angles. Students investigate, describe, and reason about decomposing and combining shapes to make other shapes. Through building, drawing, and analyzing two- and three-dimensional shapes, students develop a foundation for understanding area, volume, congruence, similarity, and symmetry in later grades.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the primary level, the big idea within mathematical modeling is showing young students how to frame, explore, and answer relevant real-world questions using mathematical ideas. A class focuses on a central open question, possibly motivated by student interest, but that then is identified, selected, or articulated by the teacher. Students gather and record data to support answering the question by using mathematics. Graphing may be done as a whole class until students are ready to try it independently. Once a visual is created, the class will decide what information or conclusions can be drawn from the representations of the information. The class might even decide they want to collect more information about this or another related idea after looking at the information (GAIMME 2016, 32).

Grade 2 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving addition and subtraction.
- Add and subtract within 20.
- Work with equal groups of objects to gain foundations for multiplication.

Number and Operations in Base Ten

- Understand place value.
- Use place value understanding and properties of operations to add and subtract.

Measurement and Data

- Measure and estimate lengths in standard units.
- Relate addition and subtraction to length.
- Work with time and money.
- Represent and interpret data.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 2 Content Standards

Operations and Algebraic Thinking (2.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving addition and subtraction.	M.2.OA.A.1	Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. <i>See Appendix, Table 1 for specific problem situations and category information.</i>
B. Add and subtract within 20.	M.2.OA.B.2	Flexibly and efficiently add and subtract within 20 using multiple mental strategies which may include counting on; making ten; decomposing a number leading to a ten; using the relationship between addition and subtraction (e.g., knowing that $8 + 4 = 12$, one knows $12 - 8 = 4$); and creating equivalent but easier or known sums (e.g., adding $6 + 7$ by creating the known equivalent $6 + 6 + 1 = 12 + 1 = 13$).
C. Work with equal groups of objects to gain foundations for multiplication.	M.2.OA.C.3	Determine whether a group of objects (up to 20) has an odd or even number of members, e.g., by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends.
	M.2.OA.C.4	Use addition to find the total number of objects arranged in rectangular arrays with up to 5 rows and up to 5 columns; write an equation to express the total as a sum of equal addends.

Number and Operations in Base Ten (2.NBT)

Cluster Statement	Notation	Standard
A. Understand place value.	M.2.NBT.A.1	Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: <ul style="list-style-type: none"> a. 100 can be thought of as a bundle of ten tens -- called a "hundred". b. The numbers 100,200,300,400,500,600,700,800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).
	M.2.NBT.A.2	Count within 1000; skip-count by 5s, 10s, and 100s.
	M.2.NBT.A.3	Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.
	M.2.NBT.A.4	Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits, and describe the result of the comparison using words and symbols ($>$, $=$, and $<$).
B. Use place value understanding and properties of operations to add and subtract.	M.2.NBT.B.5	Flexibly and efficiently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. In Grade 2, subtraction with decomposition is an exception and may include drawings/representations. ⁴
	M.2.NBT.B.6	Add up to four two-digit numbers using strategies based on place value and properties of operations.

	M.2.NBT.B.7	Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds.
	M.2.NBT.B.8	Mentally add 10 or 100 to a given number 100 - 900, and mentally subtract 10 or 100 from a given number 100 - 900.
	M.2.NBT.B.9	Explain why addition and subtraction strategies work, using place value and the properties of operations. These explanations may be supported by drawings or objects.

Measurement and Data (2.MD)

Cluster Statement	Notation	Standard
A. Measure and estimate lengths in standard units.	M.2.MD.A.1	Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
	M.2.MD.A.2	Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
	M.2.MD.A.3	Estimate lengths using units of inches, feet, centimeters, and meters.
	M.2.MD.A.4	Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.
B. Relate addition and subtraction to length	M.2.MD.B.5	Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as number lines) and equations with a symbol for the unknown number to represent the problem.
	M.2.MD.B.6	Represent whole numbers as lengths from 0 on a number line with equally spaced points corresponding to the numbers 0,1,2, ..., and represent whole-number sums and differences within 100 on a number line.
	M.2.MD.C.7	Tell and write time from analog and digital clocks to the nearest five minutes, using a.m. and p.m.

C. Work with time and money.	M.2.MD.C.8	<p>Solve word problems involving dollar bills, quarters, dimes, nickels, and pennies, using \$ and ¢ symbols appropriately.</p> <p><i>Example: If you have 2 dimes and 3 pennies, how many cents do you have?</i></p>
D. Represent and interpret data.	M.2.MD.D.9	<p>Generate measurement data by measuring lengths of several objects to the nearest whole unit, or by making repeated measurements of the same object. Show the measurements by making a line plot, where the horizontal scale is marked off in whole-number units.</p>
	M.2.MD.D.10	<p>Draw a picture graph and a bar graph (with single-unit scale) to represent a data set with up to four categories. Solve simple put together, take-apart, and compare problems using information presented in a bar graph.</p> <p><i>See Appendix, Table 1 for specific problem situations.</i></p>

Geometry (2.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.2.G.A.1	Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes. Sizes are compared directly or visually, not compared by measuring.
	M.2.G.A.2	Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.
	M.2.G.A.3	Partition circles and rectangles into two, three, or four equal shares, describe and count the shares using the words halves, thirds, and fourths, and use phrases half of, a third of, and a fourth of the whole. Describe the whole as composed of two halves, three thirds, and four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Introduction: Grade 3

In Grade 3, instructional time should focus on four critical areas: developing understanding of multiplication and division and strategies for multiplication and division within 100; developing understanding of fractions, especially unit fractions (fractions with numerator 1); developing understanding of the structure of rectangular arrays and of area; and describing and analyzing two-dimensional shapes. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models; multiplication is finding an unknown product, and division is finding an unknown factor in these situations. For equal-sized group situations, division can require finding the unknown number of groups or the unknown group size. Students use properties of operations to calculate products of whole numbers, using increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving single-digit factors. By comparing a variety of solution strategies, students learn the relationship between multiplication and division.

Students develop an understanding of fractions, beginning with unit fractions. Students view fractions in general as being built out of unit fractions, and they use fractions along with visual fraction models to represent parts of a whole. Students understand that the size of a fractional part is relative to the size of the whole. For example, $\frac{1}{2}$ of the paint in a small bucket could be less paint than $\frac{1}{3}$ of the paint in a larger bucket, but $\frac{1}{3}$ of a ribbon is longer than $\frac{1}{5}$ of the same ribbon because when the ribbon is divided into 3 equal parts, the parts are longer than when the ribbon is divided into 5 equal parts. Students are able to use fractions to represent numbers equal to, less than, and greater than one. They solve problems that involve comparing fractions by using visual fraction models and strategies based on noticing equal numerators or denominators.

Students recognize area as an attribute of two-dimensional regions. They measure the area of a shape by finding the total number of same-size units of area required to cover the shape without gaps or overlaps, a square with sides of unit length being the standard unit for measuring area. Students understand that rectangular arrays can be decomposed into identical rows or

into identical columns. By decomposing rectangles into rectangular arrays of squares, students connect area to multiplication, and justify using multiplication to determine the area of a rectangle.

Students describe, analyze, and compare properties of two-dimensional shapes. They compare and classify shapes by their sides and angles, and connect these with definitions of shapes. Students also relate their fraction work to geometry by expressing the area of part of a shape as a unit fraction of the whole.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the intermediate level, the big idea within mathematical modeling is encouraging students to move beyond the mathematical descriptions of concrete situations they experienced in the primary grades. This may include students playing a larger role in generating and defining the big modeling questions they would like to address, considering larger questions made more manageable, but still open by the teacher, determination of whether extreme cases can be captured by their model (“What If” type questions), and awareness of the value of their model in the real-world (GAIMME 2016, 33-34).

Grade 3 Overview

Operations and Algebraic Thinking

- Represent and solve problems involving multiplication and division.
- Understand properties of multiplication and the relationship between multiplication and division.
- Multiply and divide within 100.
- Solve problems involving the four operations, and identify and explain patterns in arithmetic.

Number and Operations in Base Ten

- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Develop understanding of fractions as numbers.

Measurement and Data

- Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.
- Represent and interpret data.
- Geometric measurement: understand concepts of area and relate area to multiplication and to addition.
- Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.

Geometry

- Reason with shapes and their attributes.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 3 Content Standards

Operations and Algebraic Thinking (3.OA)

Cluster Statement	Notation	Standard
A. Represent and solve problems involving multiplication and division.	M.3.OA.A.1	Interpret products of whole numbers, e.g., interpret 5×7 as the total number of objects in 5 groups of 7 objects each. <i>For example, describe a context in which a total number of objects can be expressed as 5×7.</i>
	M.3.OA.A.2	Interpret whole-number quotients of whole numbers, e.g., interpret $56 \div 8$ as the number of objects in each share when 56 objects are partitioned equally into 8 shares, or as a number of shares when 56 objects are partitioned into equal shares of 8 objects each. <i>For example, describe a context in which a number of shares or a number of groups can be expressed as $56 \div 8$.</i>
	M.3.OA.A.3	Use multiplication and division within 100 to solve word problems in situations involving equal groups, arrays, and measurement quantities, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem. <i>See Appendix, Tables 2A and 2B for specific problem situations.</i>

B. Understand properties of multiplication and the relationship between multiplication and division.	M.3.OA.B.4 [WI.2010. 3.OA.B.5]	Apply properties of operations as strategies to multiply and divide. Student use of the formal terms for these properties are not necessary. <i>Examples: If $6 \times 4 = 24$ is known, then $4 \times 6 = 24$ is also known. (Commutative property of multiplication.) $3 \times 5 \times 2$ can be found by $3 \times 5 = 15$, then $15 \times 2 = 30$, or by $5 \times 2 = 10$, then $3 \times 10 = 30$. (Associative property of multiplication.) Knowing that $8 \times 5 = 40$ and $8 \times 2 = 16$, one can find 8×7 as $8 \times (5 + 2) = (8 \times 5) + (8 \times 2) = 40 + 16 = 56$. (Distributive property.)</i>
	M.3.OA.B.5 [WI.2010. 3.OA.B.6]	Understand division as an unknown-factor problem. <i>For example, find $32 \div 8$ by finding the number that makes 32 when multiplied by 8.</i>
C. Multiply and divide within 100.	M.3.OA.C.6 [WI.2010. 3.OA.B.7]	Use multiplicative thinking to multiply and divide within 100. a. Use the meanings of multiplication and division, the relationship between the operations (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$), and properties of operations (e.g., the distributive property) to develop and understand strategies to multiply and divide within 100. b. Flexibly and efficiently use strategies, the relationship between the operations, and properties of operations to find products and quotients with multiples of 0, 1, 2, 5, & 10 within 100.
D. Solve problems involving the four operations, and identify	M.3.OA.D.7 [WI.2010. 3.OA.B.8]	Solve two-step word problems, posed with whole numbers and having whole number answers, using the four operations. Represent these problems using one or two equations with a letter standing for the unknown quantity. If one equation is used, grouping symbols (i.e. parentheses) may be needed. Assess the reasonableness of answers using mental computation and estimation strategies.

and explain patterns in arithmetic.	M.3.OA.D.8	Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations.
	[WI.2010.	<i>For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.</i>
	3.OA.B.9]	

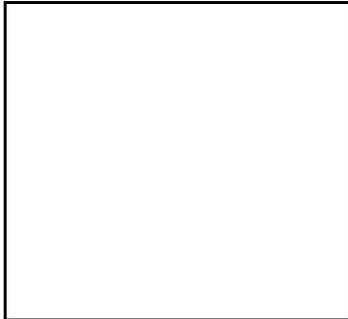
Number and Operations in Base Ten (3.NBT)

Cluster Statement	Notation	Standard
A. Use place value understanding and properties of operations to perform multi-digit arithmetic, using a variety of strategies.	M.3.NBT.A.1	Use place value understanding to generate estimates for problems in real-world situations, with whole numbers within 1,000, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates (e.g., Is my estimate too low or too high? What degree of precision do I need for this situation?).
	M.3.NBT.A.2	Flexibly and efficiently add and subtract within 1,000 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
	M.3.NBT.A.3	Multiply one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9×80 , 5×60) using strategies based on place value and properties of operations.

Number and Operations – Fractions (3.NF)

Grade 3 assessment expectations in this domain are limited to fractions with denominators 2, 3, 4, 6, and 8, but students should have instructional experiences with other sized fractions.

Cluster Statement	Notation	Standard
A. Develop understanding of fractions as numbers.	M.3.NF.A.1	<p>Understand unit fractions as the quantity formed when a whole is partitioned into equal parts, and; explain that a unit fraction is one of those parts (e.g., $1/4$).</p> <p>Understand fractions are composed of unit fractions, for example, $7/4$ is the quantity formed by 7 parts of the size $1/4$.</p>
	M.3.NF.A.2	<p>Understand and represent a fraction as a number on the number line.</p> <p>a. Understand the whole on a number line is defined as the interval from 0 to 1 and the unit fraction is defined by partitioning the interval into equal parts (i.e., equal-sized lengths).</p> <p>b. Represent fractions on a number line by iterating lengths of the unit fraction from 0. Recognize that the resulting interval represents the size of the fraction and that its endpoint locates the fraction as a number on the number line. For example, $5/3$ indicates the length of a line segment from 0 by iterating the unit fraction $1/3$ five times and its end point locates the fraction $5/3$ on the number line.</p>
	M.3.NF.A.3	<p>Explain equivalence of fractions and compare fractions by reasoning about their size.</p> <p>a. Understand two fractions as equivalent (equal) if they are the same size or name the same point on a number line.</p> <p>b. Recognize and generate simple equivalent fractions, e.g., $1/2 = 2/4$, $4/6 = 2/3$ and explain why the fractions are equivalent by using a visual fraction model (e.g., tape diagram or number line).</p>



- c. Express whole numbers as fractions ($3 = 3/1$), and recognize fractions that are equivalent to whole numbers ($4/4 = 1$).
- d. Compare two fractions with the same numerator or the same denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole. Justify the conclusions by using a visual fraction model (e.g., tape diagram or number line), and describe the result of the comparison using words and symbols ($>$, $=$, and $<$).

Measurement and Data (3.MD)

Cluster Statement	Notation	Standard
A. Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.	M.3.MD.A.1	Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line.
	M.3.MD.A.2	Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l), excluding compound units such as cm^3 and finding the geometric volume of a container. Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem. <i>See Appendix, Table 2B for problem situations. Do not include multiplicative comparison problems.</i>
B. Represent and interpret data.	M.3.MD.B.3	Draw a scaled picture graph and a scaled bar graph to represent a data set with several categories. Solve one- and two-step "how many more" and "how many less" problems using information presented in scaled bar graphs. <i>For example, draw a bar graph in which each square in the bar graph might represent 5 pets.</i>
	M.3.MD.B.4	Generate measurement data by measuring lengths using rulers marked with halves and fourths of an inch. Show the data by making a line plot, where the horizontal scale is marked off in appropriate units -- whole numbers, halves, fourths.
C. Geometric measurement: understand concepts of	M.3.MD.C.5	Recognize area as an attribute of plane figures and understand concepts of area measurement. a. A square with side length 1 unit, called "a unit square" is said to have "one square unit" of area, and can be used to measure area.

area and relate area to multiplication and to addition.		b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units.
	M.3.MD.C.6	Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units).
	M.3.MD.C.7	Relate area to the operations of multiplication and addition. <ul style="list-style-type: none"> a. Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. b. Multiply side lengths to find areas of rectangles with whole number side lengths in the context of solving real-world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. c. Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths a and $b + c$ is the sum of $a \times b$ and $a \times c$. Use area models to represent the distributive property in mathematical reasoning. d. Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real-world problems.
D. Geometric measurement: recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.	M.3.MD.D.8	Solve real-world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.

Geometry (3.G)

Cluster Statement	Notation	Standard
A. Reason with shapes and their attributes.	M.3.G.A.1	Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.
	M.3.G.A.2	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. <i>For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.</i>

Introduction: Grade 4

In Grade 4, instructional time should focus on three critical areas: developing understanding, flexibility, and efficiency with multi-digit multiplication, and developing understanding of dividing to find quotients involving multi-digit dividends; developing an understanding of fraction equivalence, addition and subtraction of fractions with like denominators, and multiplication of fractions by whole numbers; understanding that geometric figures can be analyzed and classified based on their properties, such as having parallel sides, perpendicular sides, particular angle measures, and symmetry. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students generalize their understanding of place value to 1,000,000, understanding the relative sizes of numbers in each place. They apply their understanding of models for multiplication (equal-sized groups, arrays, area models), place value, and properties of operations, in particular the distributive property, as they develop, discuss, and use efficient, accurate, and generalizable methods to compute products of multi-digit whole numbers. Depending on the numbers and the context, they select and accurately apply appropriate methods to estimate or mentally calculate products. They develop flexibility with efficient procedures for multiplying whole numbers; understand and explain why the procedures work based on place value and properties of operations; and use them to solve problems. Students apply their understanding of models for division, place value, properties of operations, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multi-digit dividends. They select and accurately apply appropriate methods to estimate and mentally calculate quotients, and interpret remainders based upon the context.

Students develop understanding of fraction equivalence and operations with fractions. They recognize that two different fractions can be equal (e.g., $15/9 = 5/3$), and they develop methods for generating and recognizing equivalent fractions. Students extend previous understandings about how fractions are built from unit fractions, composing fractions from unit fractions, decomposing fractions into unit fractions, and using the meaning of fractions and the meaning of multiplication to multiply a fraction by a whole number.

Students describe, analyze, compare, and classify two-dimensional shapes. Through building, drawing, and analyzing two-dimensional shapes, students deepen their understanding of properties of two-dimensional objects and the use of them to solve problems involving symmetry.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the intermediate level, the big idea within mathematical modeling is encouraging students to move beyond the mathematical descriptions of concrete situations they experienced in the primary grades. This may include students playing a larger role in generating and defining the big modeling questions they would like to address, considering larger questions made more manageable, but still open by the teacher, determination of whether extreme cases can be captured by their model (“What If” type questions), and awareness of the value of their model in the real-world (GAIMME 2016, 33-34).

Grade 4 Overview

Operations and Algebraic Thinking

- Use the four operations with whole numbers to solve problems.
- Gain familiarity with factors and multiples.
- Generate and analyze patterns.
- Multiply and divide within 100.

Number and Operations in Base Ten

- Generalize place value understanding for multi-digit whole numbers.
- Use place value understanding and properties of operations to perform multi-digit arithmetic.

Number and Operations—Fractions

- Extend understanding of fraction equivalence and ordering.
- Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- Understand decimal notation for fractions, and compare decimal fractions.

Measurement and Data

- Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.
- Represent and interpret data.
- Geometric measurement: understand concepts of angle and measure angles.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Geometry

- Draw and identify lines and angles, and classify shapes by properties of their lines and angles.

Grade 4 Content Standards

Operations and Algebraic Thinking (4.OA)

Cluster Statement	Notation	Standard
A. Use the four operations with whole numbers to solve problems.	M.4.OA.A.1	Interpret a multiplication equation as a multiplicative comparison, e.g., interpret $35 = 5 \times 7$ as a statement that 35 is 5 times as many as 7 and 7 times as many as 5. Represent verbal statements of multiplicative comparisons as multiplication equations.
	M.4.OA.A.2	Multiply or divide to solve word problems involving multiplicative comparison, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem, distinguishing multiplicative comparison from additive comparison. <i>See Appendix, Tables 2A & 2B.</i>
	M.4.OA.A.3	Solve multi-step word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies.

<p>B. Gain familiarity with factors and multiples.</p>	<p>M.4.OA.B.4</p>	<p>Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is prime or composite.</p>
<p>C. Generate and analyze patterns.</p>	<p>M.4.OA.C.5</p>	<p>Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.</p> <p><i>For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.</i></p>
<p>D. Multiply and divide within 100.</p>	<p>M.4.OA.D.6</p>	<p>Flexibly and efficiently multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g., knowing that $8 \times 5 = 40$, one knows $40 \div 5 = 8$) or properties of operations [e.g., knowing that 7×6 can be thought of as 7 groups of 6 so one could think 5 groups of 6 is 30 and 2 more groups of 6 is 12 and $30 + 12 = 42$ (informal use of the distributive property)].</p>

Number and Operations in Base Ten (4.NBT)

Grade 4 expectations in this domain are limited to whole numbers less than or equal to 1,000,000.

Cluster Statement	Notation	Standard
A. Generalize place value understanding for multi-digit whole numbers.	M.4.NBT.A.1	Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. <i>For example, recognize that $700 \div 70 = 10$ by applying concepts of place value and division.</i>
	M.4.NBT.A.2	Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place and describe the result of the comparison using words and symbols ($>$, $=$, and $<$.)
	M.4.NBT.A.3	Use place value understanding to generate estimates for real-world problem situations, with multi-digit whole numbers, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates. (e.g. Is my estimate too low or too high? What degree of precision do I need for this situation?)
B. Use place value understanding and properties of operations to perform multi-digit arithmetic.	M.4.NBT.B.4	Flexibly and efficiently add and subtract multi-digit whole numbers using strategies or algorithms based on place value, properties of operations, and/or the relationship between addition and subtraction.
	M.4.NBT.B.5	Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.
	M.4.NBT.B.6	Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

Number and Operations – Fractions (4.NF)

Grade 4 assessment expectations in this domain are limited to fractions with denominators 2, 3, 4, 5, 6, 8, 10, 12, and 100 but students should have instructional experiences with other sized fractions.

Cluster Statement	Notation	Standard
A. Extend understanding of fraction equivalence and ordering.	M.4.NF.A.1	<p>Understand fraction equivalence.</p> <ul style="list-style-type: none"> a. Explain why a fraction is equivalent to another fraction by using visual fraction models (e.g., tape diagrams and number lines), with attention to how the number and the size of the parts differ even though the two fractions themselves are the same size. b. Understand and use a general principle to recognize and generate equivalent fractions that name the same amount.
	M.4.NF.A.2	<p>Compare fractions with different numerators and different denominators while recognizing that comparisons are valid only when the fractions refer to the same whole. Justify the conclusions by using visual fraction models (e.g., tape diagrams and number lines) and by reasoning about the size of the fractions, using benchmark fractions (including whole numbers), or creating common denominators or numerators. Describe the result of the comparison using words and symbols ($>$, $=$, and $<$).</p>
B. Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.	M.4.NF.B.3	<p>Understand composing and decomposing fractions.</p> <ul style="list-style-type: none"> a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole. b. Decompose a fraction into a sum of unit fractions in more than one way, recording each decomposition by an equation. Justify decompositions with explanations, visual fraction models, or equations. <i>For example: $3/8 = 1/8 + 1/8 + 1/8$; $3/8 = 1/8 + 2/8$; $2\ 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8$.</i> c. Add and subtract fractions, including mixed numbers, with like denominators (e.g., $3/8 + 2/8$) and related denominators (e.g., $1/2 + 1/4$, $1/3 + 1/6$) by using visual fraction models (e.g., tape diagrams and number lines), properties of operations, and the relationship between addition and subtraction.

		<p>d. Solve word problems involving addition and subtraction of fractions with like and related denominators, including mixed numbers, by using visual fraction models and equations to represent the problem.</p> <p>Students are not required to simplify fractions to lowest terms nor use least common denominators.</p>
	M.4.NF.B.4	<p>Apply and extend previous understandings of multiplication to multiply a whole number times a fraction.</p> <p>a. Understand a fraction as a group of unit fractions or as a multiple of a unit fraction. For example, $\frac{5}{4}$ can be represented visually as 5 groups of $\frac{1}{4}$, as a sum of unit fractions $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, or as a multiple of a unit fraction $5 \times \frac{1}{4}$.</p> <p>b. Represent a whole number times a non-unit fraction (e.g., $3 \times \frac{2}{5}$) using visual fraction models and understand this as combining equal groups of the non-unit fraction (3 groups of $\frac{2}{5}$) and as a collection of unit fractions (6 groups of $\frac{1}{5}$), recognizing this product as $\frac{6}{5}$.</p> <p>c. Solve word problems involving multiplication of a whole number times a fraction by using visual fraction models and equations to represent the problem. Understand a reasonable answer range when multiplying with fractions.</p>
C. Understand decimal notation for fractions, and compare decimal fractions.	M.4.NF.C.5	<p>Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.</p> <p><i>For example, express $\frac{3}{10}$ as $\frac{30}{100}$, and add $\frac{3}{10} + \frac{4}{100} = \frac{34}{100}$.</i></p>
	M.4.NF.C.6	<p>Use decimal notation for fractions with denominators 10 or 100, connect decimals to real-world contexts, and represent with visual models (e.g., number line or area model).</p> <p><i>For example, rewrite 0.62 as $\frac{62}{100}$; describe a length as 0.62 meters; locate 0.62 on a number line.</i></p>
	M.4.NF.C.7	<p>Compare decimals to hundredths by reasoning about their size and using benchmarks. Recognize that comparisons are valid only when the decimals refer to the same whole. Justify the conclusions, by using explanations or visual models (e.g., number line or area model) and describe the result of the comparison using words and symbols ($>$, $=$, and $<$).</p>

Measurement and Data (4.MD)

Cluster Statement	Notation	Standard
A. Solve problems involving measurement and conversion of measurements from a larger unit to a smaller unit.	M.4.MD.A.1	<p>Know relative sizes of measurement units within one system of units including km, m, cm; kg, g; lb., oz.; l, ml; hr, min, sec. Within a single system of measurement, express measurements in a larger unit in terms of a smaller unit. Record measurement equivalents in a two-column table.</p> <p><i>For example, know that 1 ft is 12 times as long as 1 in. Express the length of a 4 ft snake as 48 in. Generate a conversion table for feet and inches listing the number pairs (1, 12), (2, 24), (3, 36), ...</i></p>
	M.4.MD.A.2	<p>Use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money, including problems involving simple fractions or decimals, and problems that require expressing measurements given in a larger unit in terms of a smaller unit. Represent measurement quantities using diagrams such as a number line that feature a measurement scale.</p>
	M.4.MD.A.3	<p>Apply the area and perimeter formulas for rectangles in real-world and mathematical problems.</p> <p><i>For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor.</i></p>
B. Represent and interpret data.	M.4.MD.B.4	<p>Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Solve problems involving addition and subtraction of fractions by using information presented in line plots.</p> <p><i>For example, from a line plot find and interpret the difference in length between the longest and shortest specimens in an insect collection.</i></p>
C. Geometric measurement: understand concepts	M.4.MD.C.5	<p>Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:</p> <ol style="list-style-type: none"> a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays

of angle and measure angles.		intersect the circle. An angle that turns through $\frac{1}{360}$ of a circle is called a "one-degree angle" and can be used to measure angles. b. An angle that turns through n one-degree angles is said to have an angle measure of n degrees.
	M.4.MD.C.6	Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
	M.4.MD.C.7	Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure.

Geometry (4.G)

Cluster Statement	Notation	Standard
A. Draw and identify lines and angles, and classify shapes by properties of their lines and angles.	M.4.G.A.1	Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.
	M.4.G.A.2	Classify two-dimensional figures based on the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size. Recognize right triangles as a category, and identify right triangles.
	M.4.G.A.3	Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.

Introduction: Grade 5

In Grade 5, instructional time should focus on three critical areas: developing flexible strategies with addition and subtraction of fractions, and developing understanding of the multiplication of fractions and of division of fractions in limited cases (unit fractions divided by whole numbers and whole numbers divided by unit fractions); extending division to 2-digit divisors, integrating decimal fractions into the place value system and developing understanding of operations with decimals to hundredths, and developing flexibility with strategies to compute whole number and decimal operations; and developing understanding of volume. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students apply their understanding of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They develop flexible strategies in calculating sums and differences of fractions, and make reasonable estimates of them. Students also use the meaning of fractions, of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for multiplying and dividing fractions make sense. (Note: this is limited to the case of dividing unit fractions by whole numbers and whole numbers by unit fractions.)

Students develop an understanding of why division procedures work based on the meaning of base-ten numerals and properties of operations. They compute efficiently with multi-digit addition, subtraction, multiplication, and division while flexibly applying strategies. They apply their understandings of models for decimals, decimal notation, and properties of operations to add and subtract decimals to hundredths. They develop flexible strategies in these computations, and make reasonable estimates of their results. Students use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain why the procedures for multiplying and dividing finite decimals make sense. They compute products and quotients of decimals to hundredths efficiently and accurately.

Students recognize volume as an attribute of three-dimensional space. They understand that volume can be measured by finding the total number of same-size units of volume required to fill the space without gaps or overlaps. They understand that a 1-unit by 1-unit by 1-unit cube is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating and measuring volume. They decompose three-dimensional shapes and find volumes of right rectangular prisms by viewing them as decomposed into layers of arrays of cubes. They measure necessary attributes of shapes in order to determine volumes to solve real-world and mathematical problems.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the intermediate level, the big idea within mathematical modeling is encouraging students to move beyond the mathematical descriptions of concrete situations they experienced in the primary grades. This may include students playing a larger role in generating and defining the big modeling questions they would like to address, considering larger questions made more manageable, but still open by the teacher, determination of whether extreme cases can be captured by their model (“What If” type questions), and awareness of the value of their model in the real-world (GAIMME 2016, 33-34).

Grade 5 Overview

Operations and Algebraic Thinking

- Write and interpret numerical expressions.
- Analyze patterns and relationships.

Number and Operations in Base Ten

- Understand the place value system.
- Perform operations with multi-digit whole numbers and with decimals to hundredths.

Number and Operations—Fractions

- Use equivalent fractions as a strategy to add and subtract fractions.
- Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

Measurement and Data

- Convert like measurement units within a given measurement system.
- Represent and interpret data.
- Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.

Geometry

- Graph points on the coordinate plane to solve real-world and mathematical problems.
- Classify two-dimensional figures into categories based on their properties.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 5 Content Standards

Operations and Algebraic Thinking (5.OA)

Cluster Statement	Notation	Standard
A. Write and interpret numerical expressions.	M.5.OA.A.1	Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.
	M.5.OA.A.2	Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them. <i>For example, express the calculation "add 8 and 7, then multiply by 2" as $2 \times (8 + 7)$. Recognize that $3 \times (18932 + 921)$ is three times as large as $18932 + 921$, without having to calculate the indicated sum or product.</i>
B. Analyze patterns and relationships.	M.5.OA.B.3	Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. <i>For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.</i>

Number and Operations in Base Ten (5.NBT)

Cluster Statement	Notation	Standard
A. Understand the place value system.	M.5.NBT.A.1	Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left.
	M.5.NBT.A.2	Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10.
	M.5.NBT.A.3	Read, write, and compare decimals to thousandths. <ul style="list-style-type: none"> a. Read and write decimals to thousandths using base-ten numerals, number names, and expanded form, e.g., $347.392 = 3 \times 100 + 4 \times 10 + 7 \times 1 + 3 \times (1/10) + 9 \times (1/100) + 2 \times (1/1000)$. b. Compare decimals to thousandths based on meanings of the digits in each place and describe the result of the comparison using words and symbols ($>$, $=$, and $<$).
	M.5.NBT.A.4	Use place value understanding to generate estimates for problems in real-world situations, with decimals, using strategies such as mental math, benchmark numbers, compatible numbers, and rounding. Assess the reasonableness of their estimates (e.g. Is my estimate too low or too high? What degree of precision do I need for this situation?)
B. Perform operations with multi-digit whole numbers and with decimals to hundredths.	M.5.NBT.B.5	Flexibly and efficiently multiply multi-digit whole numbers using strategies or algorithms based on place value, area models, and the properties of operations.
	M.5.NBT.B.6	Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.

	M.5.NBT.B.7	Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used.
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Number and Operations – Fractions (5.NF)

Students are not required to simplify fractions to lowest terms nor use least common denominators.

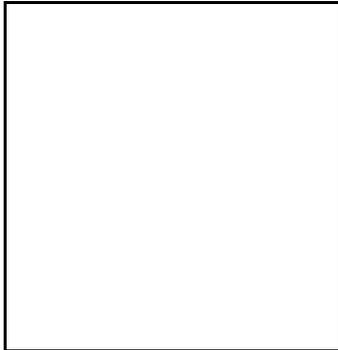
Cluster Statement	Notation	Standard
A. Use equivalent fractions as a strategy to add and subtract fractions.	M.5.NF.A.1	Add and subtract fractions and mixed numbers using flexible and efficient strategies, including renaming fractions with equivalent fractions. Justify using visual models (e.g., tape diagrams or number lines) and equations. <i>For example, $2/3 + 5/4 = 8/12 + 15/12 = 23/12$.</i>
	M.5.NF.A.2	Solve word problems involving addition and subtraction of fractions referring to the same whole using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. <i>For example, recognize an incorrect result $2/5 + 1/2 = 3/7$, by observing that $3/7 < 1/2$.</i>
B. Apply and extend previous understandings of multiplication and division to multiply and divide fractions.	M.5.NF.B.3	Interpret a fraction as an equal sharing division situation, where a quantity (the numerator) is divided into equal parts (the denominator). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, by using visual fraction models (e.g., tape diagrams or area models) or equations to represent the problem. <i>For example, when 3 wholes are shared equally among 4 people each person has a share of size $3/4$. If 9 people want to share a 50-pound sack of rice equally by weight, how many pounds of rice should each person get? Between what two whole numbers does your answer lie?</i>
	M.5.NF.B.4	Apply and extend previous understandings of multiplication to multiply a fraction times a whole number (e.g., $2/3 \times 4$) or a fraction times a fraction (e.g., $2/3 \times 4/5$), including mixed numbers. a. Represent word problems involving multiplication of fractions using visual models to develop flexible and efficient strategies. <i>For example, use a visual fraction model to show $(2/3) \times 4 = 8/3$, and create a story context for this equation. Do the same with $(2/3) \times (4/5) = 8/15$.</i>

		<p>b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas.</p>
	M.5.NF.B.5	<p>Interpret multiplication as scaling (resizing) by estimating whether a product will be larger or smaller than a given factor on the basis of the size of the other factor, without performing the indicated multiplication.</p> <p>a. Explain why multiplying a given number by a fraction greater than 1 results in a product greater than the given number and explain why multiplying a given number by a fraction less than 1 results in a product smaller than the given number.</p> <p>b. Relate the principle of fraction equivalence to the effect of multiplying or dividing a fraction by 1 or an equivalent form of 1 (e.g., $\frac{3}{3}$, $\frac{5}{5}$).</p>
	M.5.NF.B.6	<p>Solve real-world problems involving multiplication of fractions and mixed numbers by using visual fraction models (e.g., tape diagrams, area models, or number lines) and equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers.</p>
	M.5.NF.B.7	<p>Apply and extend previous understandings of division to divide unit fractions by whole numbers (e.g., $\frac{1}{3} \div 4$) and whole numbers by unit fractions (e.g., $4 \div \frac{1}{5}$).</p> <p>Students able to multiply fractions in general can develop strategies to divide fractions in general, by reasoning about the relationship between multiplication and division. But division of a fraction by a fraction is not a requirement at this grade.</p> <p>a. Interpret and represent division of a unit fraction by a non-zero whole number as an equal sharing division situation.</p> <p><i>For example, create a story context for $(\frac{1}{3}) \div 4$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $(\frac{1}{3}) \div 4 = \frac{1}{12}$ because $(\frac{1}{12}) \times 4 = \frac{1}{3}$.</i></p> <p>b. Interpret and represent division of a whole number by a unit fraction as a measurement division situation.</p>

		<p><i>For example, create a story context for $4 \div (1/5)$, and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that $4 \div (1/5) = 20$ because $20 \times (1/5) = 4$.</i></p> <p>c. Solve real-world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions by using visual fraction models and equations to represent the problem.</p> <p><i>For example, how much chocolate will each person get if 4 people share $1/3$ lb. of chocolate equally? Each person gets $1/12$ lb. of chocolate. How many $1/5$-cup servings are in 4 cups of raisins? There are 20 servings of size $1/5$-cup of raisins.</i></p>
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Measurement and Data (5.MD)

Cluster Statement	Notation	Standard
A. Convert like measurement units within a given measurement system.	M.5.MD.A.1	Convert among different-sized standard measurement units within a given measurement system (e.g., convert 5 cm to 0.05 m), and use these conversions in solving multi-step, real-world problems.
B. Represent and interpret data.	M.5.MD.B.2	<p>Make a line plot to display a data set of measurements in fractions of a unit ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$). Use operations on fractions for this grade to solve problems involving information presented in line plots.</p> <p><i>For example, given different measurements of liquid in identical beakers, find the amount of liquid each beaker would contain if the total amount in all the beakers were redistributed equally.</i></p>
C. Geometric measurement: understand concepts of volume and relate volume to multiplication and to addition.	M.5.MD.C.3	<p>Recognize volume as an attribute of solid figures and understand concepts of volume measurement.</p> <ul style="list-style-type: none"> a. A cube with side length 1 unit, called a "unit cube", is said to have "one cubic unit" of volume, and can be used to measure volume. b. A solid figure which can be packed without gaps or overlaps using n unit cubes is said to have a volume of n cubic units.
	M.5.MD.C.4	Measure volumes by counting unit cubes, using cubic cm, cubic in, cubic ft, and improvised units.
	M.5.MD.C.5	<p>Relate volume to the operations of multiplication and addition and solve real-world and mathematical problems involving volume.</p> <ul style="list-style-type: none"> a. Find the volume of a right rectangular prism with whole-number side lengths by packing it with unit cubes, and show that the volume is the same as would be found by multiplying the edge



lengths, equivalently by multiplying the height by the area of the base. Represent threefold whole-number products as volumes, e.g., to represent the associative property of multiplication.

- b. Apply the formulas $V = l \times w \times h$ and $V = b \times h$ for rectangular prisms to find volumes of right rectangular prisms with whole number edge lengths in the context of solving real-world and mathematical problems.
- c. Recognize volume as additive. Find volumes of solid figures composed of two non-overlapping right rectangular prisms by adding the volumes of the non-overlapping parts, applying this technique to solve real-world problems.

Geometry (5.G)

Cluster Statement	Notation	Standard
A. Graph points on the coordinate plane to solve real-world and mathematical problems.	M.5.G.A.1	Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., <i>x-axis</i> and <i>x-coordinate</i> , <i>y-axis</i> and <i>y-coordinate</i>).
	M.5.G.A.2	Represent real-world and mathematical problems by graphing points in the first quadrant of the coordinate plane, and interpret coordinate values of points in the context of the situation.
B. Classify two-dimensional figures into categories based on their properties.	M.5.G.B.3	Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. <i>For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.</i>
	M.5.G.B.4	Classify two-dimensional figures in a hierarchy based on properties.

6-8 Middle School Standards

This section includes 6-8 specific Standards for Mathematical Practice as well as grade level introductions, overviews and Standards for Mathematical Content for each grade, six through eight.

6-8 Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

6-8 Mathematically proficient middle school students set out to understand a problem and then look for entry points to its solution. Students identify questions to ask and make observations about the situation by using strategies such as noticing and wondering. Students make assumptions where needed to make the problem more clearly defined. They analyze problem conditions and goals, translating, for example, verbal descriptions into equations, diagrams, or graphs as part of the process. They consider analogous problems, and try special cases and simpler forms of the original in order to gain insight into its solution. For example, to understand why a 20% discount followed by a 20% markup does not return an item to its original price, they might translate the situation into a tape diagram or a general equation; or they might first consider the situation for an item priced at \$100.

Mathematically proficient students can explain how alternate representations of problem conditions relate to each other. For example, they can identify connections between the solution to a word problem that uses only arithmetic and a solution that uses variables and algebra; and they can navigate among verbal descriptions, tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change.

Mathematically proficient students check their approach, continually asking themselves “Does this approach make sense?” and “Can I solve the problem in a different way?” Students ask themselves these types of questions as a way to persevere through problem solving. While working on a problem, they monitor and evaluate their progress and change course if necessary. Students will reflect and revise their solution as needed. They can understand the approaches of others to solving complex problems and compare approaches.

Math Practice 2: Reason abstractly and quantitatively.

6-8 Mathematically proficient middle school students make sense of quantities and relationships in problem situations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. They represent problem situations using symbols and then manipulate those symbols in search of a solution (decontextualize). They can, for example, solve problems involving unit rates by representing the situations in equation form. Mathematically proficient students also pause as needed during problem solving to validate the meaning of the symbols involved. In the process, they can look back at the applicable units of measure

to clarify or inform solution pathways (contextualize). Students can integrate quantitative information and concepts expressed in text and visual formats. Students can examine the constant and coefficient used in a linear function and express the meaning of those numbers related to a contextual situation. They can work with the function in different representations, such as a graph, keeping in mind the slope and vertical intercept have meaning related to the context. Quantitative reasoning also entails knowing and flexibly using different properties of operations and objects. For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others. (Gutierrez 2017)

6-8 Mathematically proficient middle school students understand and use assumptions, definitions, and previously established results in constructing verbal and written arguments. They understand the importance of making and exploring the validity of conjectures. They can recognize and appreciate the use of counterexamples. For example, using numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that $5 - 2x$ is equivalent to $3x$. Conversely, given a pair of equivalent algebraic expressions, they can show that the two expressions name the same number regardless of which value is substituted into them by showing which properties of operations can be applied to transform one expression into the other. They can explain and justify their conclusions to others using numerals, symbols, and visuals; in turn they can listen or read others' arguments, deciding whether they make sense and asking questions to clarify the arguments. They also reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example, they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explain what it is. Students engage in collaborative discussions, drawing on evidence from problem texts and arguments of others, follow conventions for collegial discussions, and qualify their own views in light of evidence presented. They consider questions such as "How did you get that?" "Why is that true?" and "Does that always work?" They can present their findings and results to a given audience through a variety of formats such as posters, whiteboards, and interactive materials.

Math Practice 4: Model with mathematics.

6-8 Mathematically proficient middle school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This might be as simple as translating a verbal or written description to a drawing or mathematical expression. It might entail using the mathematics of proportional relationships to plan a school event, or using data to analyze a problem in the community. Students that engage in modeling have choice when solving problems. Mathematically proficient students are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. For example, they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability. They are able to identify important quantities in a given relationship such as rates of change and represent situations using such tools as diagrams, tables, graphs,

flowcharts and formulas. They can analyze their representations mathematically, use the results in the context of the situation, and then reflect on whether the results make sense. Throughout the process students are able to refine and extend their model in order to improve them.

Note: Although physical objects and drawings can be used to model a situation, using these tools absent a contextual situation is not an example of practice standard MP.4. For example, drawing an area model to illustrate the distributive property in $4(t + s) = 4t + 4s$ would not be an example of practice standard MP.4. Practice standard MP.4 is about applying math to a problem in context.

Math Practice 5: Use appropriate tools strategically.

6-8 Mathematically proficient middle school students strategically consider the available tools when solving a mathematical problem or exploring a mathematical relationship. These tools might include pencil and paper, concrete models, a ruler, a protractor, a graphing tool, a spreadsheet, a statistical package, or dynamic geometry software. Proficient students make sound decisions about when each of these tools might be helpful, recognizing both the insights to be gained and their limitations. For example, they use estimation to check reasonableness; graph functions defined by expressions to picture the way one quantity depends on another; use algebra tiles to see how the properties of operations familiar from the elementary grades continue to apply to algebraic expressions; use an area model to visualize multiplication of rational numbers; use graphing calculators to approximate solutions to systems of equations; use spreadsheets to analyze data sets of realistic size; or use dynamic geometry software to discover properties of parallelograms. Students are also strategic about when not to use tools, such as by simplifying an expression before substituting values into it (MP.7), or rounding the inputs to a calculation and calculating on paper when an approximate answer is enough (MP.6). When making mathematical models, students know that technology can enable them to visualize the results of their assumptions, to explore consequences, and to compare predictions with data (MP.4). Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

Math Practice 6: Attend to precision.

6-8 Mathematically proficient middle school students communicate precisely to others both verbally and in writing. They present claims and findings, emphasizing salient points in a focused, coherent manner with relevant evidence, sound and valid reasoning, well-chosen details, and precise language. They use clear definitions in discussion with others and in their own reasoning and determine the meaning of symbols, terms, and phrases as used in specific mathematical contexts. For example, they can use the definition of rational numbers to explain why the $\sqrt{2}$ is irrational and describe congruence and similarity in terms of transformations in the plane. They decide which parts of a problem need to be defined by a variable, state the meaning of the symbols, consistently and appropriately, such as independent and dependent variables. They are careful about specifying units of measure, and label axes to display the correct correspondence between quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate to the context. For example, they accurately apply scientific notation to large numbers and use measures of center to describe data sets. Diligence and attention

to detail are mathematical virtues. Mathematically proficient students care that an answer is right or reasonable; they attend to precision when they check their work; they solve the problem another way; they take responsibility for mistakes and correct them.

Math Practice 7: Look for and make use of structure.

6-8 Mathematically proficient middle school students look closely to discern a pattern or structure. They might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation $3x = 2y$ represents a proportional relationship with a unit rate of $3/2 = 1.5$. They might recognize how the Pythagorean Theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism. They also can step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by six. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines or understanding $1.05a$ as an original value, a , plus 5% of that value, $0.05a$. They can evaluate numeric expressions without combining each term in the order they are given. Example $2\frac{7}{8} - (-31/7) - 2\frac{7}{8}$. Students use structure within these calculations, so they work slow to fast and purposefully.

Math Practice 8: Look for and express regularity in repeated reasoning.

6-8 Mathematically proficient middle school students notice if calculations are repeated, and look for both general methods and general and efficient methods. Working with tables of equivalent ratios, they might deduce the corresponding multiplicative relationships and make generalizations about the relationship to rates. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity with which interior angle sums increase with the number of sides in a polygon might lead them to the general formula for the interior angle sum of an n -gon. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their results throughout all stages of the process.

Introduction: Grade 6

In Grade 6, instructional time should focus on four critical areas: connecting ratio and rate to whole number multiplication and division and using concepts of ratio and rate to solve problems; completing understanding of division of fractions and extending the notion of number to the system of rational numbers, which includes negative numbers; writing, interpreting, and using expressions and equations; and developing understanding of statistical thinking. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students use reasoning about multiplication and division to solve ratio and rate problems about quantities. By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative size of quantities, students connect their understanding of multiplication and division with ratios and rates. Thus, students expand the scope of problems for which they can use multiplication and division to solve problems, and they connect ratios and fractions. Students solve a wide variety of problems involving ratios and rates.

Students use the meaning of fractions, the meanings of multiplication and division, and the relationship between multiplication and division to understand and explain why the procedures for dividing fractions make sense. Students use these operations to solve problems. Students extend their previous understandings of number and the ordering of numbers to the full system of rational numbers, which includes negative rational numbers, and in particular negative integers. They reason about the order and absolute value of rational numbers and about the location of points in all four quadrants of the coordinate plane.

Students understand the use of variables in mathematical expressions. They write expressions and equations that correspond to given situations, evaluate expressions, and use expressions and formulas to solve problems. Students understand that expressions in different forms can be equivalent, and they use the properties of operations to rewrite expressions in equivalent forms. Students know that the solutions of an equation are the values of the variables that make the equation true. Students use

properties of operations and the idea of maintaining the equality of both sides of an equation to solve simple one-step equations. Students construct and analyze tables, such as tables of quantities that are in equivalent ratios, and they use equations (such as $3x = y$) to describe relationships between quantities.

Building on and reinforcing their understanding of number, students begin to develop their ability to think statistically. Students recognize that a data distribution may not have a definite center and that different ways to measure center yield different values. The median measures center in the sense that it is roughly the middle value. The mean measures center in the sense that it is the value that each data point would take on if the total of the data values were redistributed equally, and also in the sense that it is a balance point. Students recognize that a measure of variability (interquartile range or mean absolute deviation) can also be useful for summarizing data because two very different sets of data can have the same mean and median yet be distinguished by their variability.

Students learn to describe and summarize numerical data sets, identifying clusters, peaks, gaps, and symmetry, considering the context in which the data were collected. Students in Grade 6 also build on their work with area in elementary school by reasoning about relationships among shapes to determine area, surface area, and volume. They find areas of right triangles, other triangles, and special quadrilaterals by decomposing these shapes, rearranging or removing pieces, and relating the shapes to rectangles. Using these methods, students discuss, develop, and justify formulas for areas of triangles and parallelograms. Students find areas of polygons and surface areas of prisms and pyramids by decomposing them into pieces whose area they can determine. They reason about right rectangular prisms with fractional side lengths to extend formulas for the volume of a right rectangular prism to fractional side lengths. They prepare for work on scale drawings and constructions in Grade 7 by drawing polygons in the coordinate plane.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the middle school level, the big idea within mathematical modeling is encouraging students to act more independently as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (GAIMME 2016, 34).

Grade 6 Overview

Ratios and Proportional Relationships

- Understand ratio concepts and use ratio reasoning to solve problems.

The Number System

- Apply and extend previous understandings of multiplication and division to divide fractions by fractions.
- Flexibly and efficiently compute with multi-digit numbers and find common factors and multiples.
- Apply and extend previous understandings of numbers to the system of rational numbers.

Expressions and Equations

- Apply and extend previous understandings of arithmetic to algebraic expressions.
- Reason about and solve one-variable equations and inequalities.
- Represent and analyze quantitative relationships between dependent and independent variables.

Geometry

- Solve real-world and mathematical problems involving area, surface area, and volume.

Statistics and Probability

- Develop understanding of statistical variability.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

- Summarize and describe distributions.

Grade 6 Content Standards

Ratios and Proportional Relationships (6.RP)

Cluster Statement	Notation	Standard
A. Understand ratio concepts and use ratio reasoning to solve problems.	M.6.RP.A.1	<p>Understand the concept of a ratio and use ratio language to describe a ratio relationship between two quantities.</p> <p><i>For example, "The ratio of wings to beaks in the bird house at the zoo was 2:1, because for every 2 wings there was 1 beak." "For every vote candidate A received, candidate C received nearly three votes."</i></p>
	M.6.RP.A.2	<p>Understand the concept of a unit rate a/b associated with a ratio $a:b$ with $b \neq 0$, and use rate language in the context of a ratio relationship.</p> <p><i>For example, "This recipe has a ratio of 3 cups of flour to 4 cups of sugar, so there is $3/4$ cup of flour for each cup of sugar." "We paid \$75 for 15 hamburgers, which is a rate of \$5 per hamburger."</i></p> <p>Expectations for unit rates in this grade are limited to non-complex fractions.</p>
	M.6.RP.A.3	<p>Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.</p> <ol style="list-style-type: none"> Make tables of equivalent ratios relating quantities with whole number measurements, find missing values in the tables, and plot the pairs of values on the coordinate plane. Use tables to compare ratios. Solve unit rate problems including those involving unit pricing and constant speed. For example, if it took 7 hours to mow 4 lawns, then at that rate, how many lawns could be mowed in 35 hours? At what rate were lawns being mowed?



- c. Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.
- d. Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities.

The Number System (6.NS)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of multiplication and division to divide fractions by fractions.	M.6.NS.A.1	<p>Interpret, represent and compute division of fractions by fractions; and solve word problems by using visual fraction models (e.g., tape diagrams, area models, or number lines), equations, and the relationship between multiplication and division.</p> <p><i>For example, create a story context for $(2/3) \div (3/4)$ such as “How many $3/4$-cup servings are in $2/3$ of a cup of yogurt” or “How wide is a rectangular strip of land with length $3/4$ mile and area $2/3$ square mile?” Explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$.</i></p>
B. Flexibly and efficiently compute with multi-digit numbers and find common factors and multiples.	M.6.NS.B.2	Flexibly and efficiently divide multi-digit whole numbers using strategies or algorithms based on place value, area models, and the properties of operations.
	M.6.NS.B.3	Flexibly and efficiently add, subtract, multiply, and divide multi-digit decimals using strategies or algorithms based on place value, visual models, the relationship between operations and the properties of operations.
	M.6.NS.B.4	<p>Find the greatest common factor of two whole numbers less than or equal to 100 and the least common multiple of two whole numbers less than or equal to 12. Use the distributive property to express a sum of two whole numbers 1-100 with a common factor as a multiple of a sum of two whole numbers with no common factor.</p> <p><i>For example, express $36 + 8$ as $4(9 + 2)$.</i></p>
C. Apply and extend previous understandings	M.6.NS.C.5	Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation.

<p>of numbers to the system of rational numbers.</p>	M.6.NS.C.6	<p>Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.</p> <ol style="list-style-type: none"> Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$, and that 0 is its own opposite. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes. Find and position integers and other rational numbers on a horizontal or vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.
	M.6.NS.C.7	<p>Understand ordering and absolute value of rational numbers.</p> <ol style="list-style-type: none"> Interpret statements of inequality as statements about the relative position of two numbers on a number line diagram. For example, interpret $-3 > -7$ as a statement that -3 is located to the right of -7 on a number line oriented from left to right. Write, interpret, and explain statements of order for rational numbers in real-world contexts. For example, write $-3\text{ }^{\circ}\text{C} > -7\text{ }^{\circ}\text{C}$ to express the fact that $-3\text{ }^{\circ}\text{C}$ is warmer than $-7\text{ }^{\circ}\text{C}$. Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. For example, for an account balance of -30 dollars, write $-30 = 30$ to describe the size of the debt in dollars. Distinguish comparisons of absolute value from statements about order. For example, recognize that an account balance less than -30 dollars represents a debt greater than 30 dollars.

	M.6.NS.C.8	Solve real-world and mathematical problems by graphing points in all four quadrants of the coordinate plane. Include use of coordinates and absolute value to find distances between points with the same first coordinate or the same second coordinate.
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The Expressions and Equations (6.EE)

Cluster Statement	Notation	Standard
A. Apply and extend previous understandings of arithmetic to algebraic expressions	M.6.EE.A.1	Write and evaluate numerical expressions involving whole-number exponents.
	M.6.EE.A.2	<p>Write, read, and evaluate expressions in which letters stand for numbers.</p> <p>a. Write expressions that record operations with numbers and with letters standing for numbers.</p> <p><i>For example, express the calculation "Subtract y from 5" as $5 - y$.</i></p> <p>b. Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity.</p> <p><i>For example, describe the expression $2(8 + 7)$ as a product of two factors; view $(8 + 7)$ as both a single entity and a sum of two terms.</i></p> <p>c. Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations).</p> <p><i>For example, use the formulas $V = s^3$ and $A = 6s^2$ to find the volume and surface area of a cube with sides of length $s = 1/2$.</i></p>
	M.6.EE.A.3	Apply the properties of operations to generate equivalent expressions.

		<p><i>For example, apply the distributive property to the expression $3(2 + x)$ to produce the equivalent expression $6 + 3x$; apply the distributive property to the expression $24x + 18y$ to produce the equivalent expression $6(4x + 3y)$; apply properties of operations to $y + y + y$ to produce the equivalent expression $3y$.</i></p>
	M.6.EE.A.4	<p>Identify when two expressions are equivalent (e.g., when the two expressions name the same number regardless of which value is substituted into them).</p> <p><i>For example, the expressions $y + y + y$ and $3y$ are equivalent because they name the same number regardless of which number y stands for.</i></p>
B. Reason about and solve one-variable equations and inequalities.	M.6.EE.B.5	Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.
	M.6.EE.B.6	Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.
	M.6.EE.B.7	Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p , q and x are all nonnegative rational numbers.
	M.6.EE.B.8	Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.
C. Represent and analyze quantitative relationships between	M.6.EE.C.9	Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

dependent and independent variables.		<i>For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.</i>
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Geometry (6.G)

Cluster Statement	Notation	Standard
A. Solve real-world and mathematical problems involving area, surface area, and volume.	M.6.G.A.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
	M.6.G.A.2	Find volumes of right rectangular prisms with fractional edge lengths by using physical or virtual unit cubes. Develop (construct) and apply the formulas $V = lwh$ and $V = Bh$ to find volumes of right rectangular prisms in the context of solving real-world and mathematical problems.
	M.6.G.A.3	Draw polygons in the coordinate plane given coordinates for the vertices; use coordinates to find the length of a side joining points with the same first coordinate or the same second coordinate. Apply these techniques in the context of solving real-world and mathematical problems.
	M.6.G.A.4	Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.

Statistics and Probability (6.SP)

Cluster Statement	Notation	Standard
A. Develop understanding of statistical variability.	M.6.SP.A.1	Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers. <i>For example, "How old am I?" is not a statistical question, but "How old are the students in my school?" is a statistical question because one anticipates variability in students' ages.</i>
	M.6.SP.A.2	Understand that a set of data collected to answer a statistical question has a distribution which can be described by its center, spread, and overall shape.
	M.6.SP.A.3	Recognize that a measure of center for a numerical data set summarizes all of its values with a single number, while a measure of variation describes how its values vary with a single number.
B. Summarize and describe distributions.	M.6.SP.B.4	Display numerical data in plots on a number line, including dot plots, histograms, and box plots.
	M.6.SP.B.5	Summarize numerical data sets in relation to their context, such as by: <ol style="list-style-type: none"> a. Reporting the number of observations. b. Describing the nature of the attribute under investigation, including how it was measured and its units of measurement. c. Describing any overall pattern and any striking deviations from the overall pattern with reference to the context in which the data were gathered and the quantitative measures of center (median and/or mean) and variability (interquartile range and/or mean absolute deviation) were given. d. Relating the choice of measures of center and variability to the shape of the data distribution and the context in which the data were gathered.

Introduction Grade 7

In Grade 7, instructional time should focus on four critical areas: developing understanding of and applying proportional relationships; developing understanding of operations with rational numbers and working with expressions and linear equations; solving problems involving scale drawings and informal geometric constructions, and working with two- and three-dimensional shapes to solve problems involving area, surface area, and volume; and drawing inferences about populations based on samples. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the *Standards for Mathematical Practice*. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students extend their understanding of ratios and develop understanding of proportionality to solve single- and multi-step problems. Students use their understanding of ratios and proportionality to solve a wide variety of percent problems, including those involving discounts, interest, taxes, tips, and percent increase or decrease. Students solve problems about scale drawings by relating corresponding lengths between the objects or by using the fact that relationships of lengths within an object are preserved in similar objects. Students graph proportional relationships and understand the unit rate informally as a measure of the steepness of the related line, called the slope. They distinguish proportional relationships from other relationships.

Students develop a unified understanding of number, recognizing fractions, decimals (that have a finite or a repeating decimal representation), and percents as different representations of rational numbers. Students extend addition, subtraction, multiplication, and division to all rational numbers, maintaining the properties of operations and the relationships between addition and subtraction, and multiplication and division. By applying these properties, and by viewing negative numbers in terms of everyday contexts (e.g., amounts owed or temperatures below zero), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. They use the arithmetic of rational numbers as they formulate expressions and equations in one variable and use these equations to solve problems.

Students continue their work with area from Grade 6, solving problems involving the area and circumference of a circle and surface area of three-dimensional objects. In preparation for work on congruence and similarity in Grade 8 they reason about

relationships among two-dimensional figures using scale drawings and informal geometric constructions, and they gain familiarity with the relationships between angles formed by intersecting lines. Students work with three-dimensional figures, relating them to two-dimensional figures by examining cross-sections. They solve real-world and mathematical problems involving area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes and right prisms.

Students build on their previous work with single data distributions to compare two data distributions and address questions about differences between populations. They begin informal work with random sampling to generate data sets and learn about the importance of representative samples for drawing inferences.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the middle school level, the big idea within mathematical modeling is encouraging students to act more independently as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (GAIMME 2016, 34).

Grade 7 Overview

Ratios and Proportional Relationships

- Analyze proportional relationships and use them to solve real-world and mathematical problems.

The Number System

- Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

Expressions and Equations

- Use properties of operations to generate equivalent expressions.
- Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

Geometry

- Draw, construct and describe geometrical figures and describe the relationships between them.
- Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

Statistics and Probability

- Use random sampling to draw inferences about a population.
- Draw informal comparative inferences about two populations.
- Investigate chance processes and develop, use, and evaluate probability models.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 7 Content Standards

Ratios and Proportional Relationships (7.RP)

Cluster Statement	Notation	Standard
A. Analyze proportional relationships and use them to solve real-world and mathematical problems.	M.7.RP.A.1	Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units. <i>For example, if a person walks $\frac{1}{2}$ mile in each $\frac{1}{4}$ hour, compute the unit rate as the complex fraction $\frac{1/2}{1/4}$ miles per hour, equivalently 2 miles per hour.</i>
	M.7.RP.A.2	Recognize and represent proportional relationships between quantities. <ol style="list-style-type: none"> Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships. Represent proportional relationships by equations. <i>For example, if total cost t is proportional to the number n of items purchased at a constant price p, the relationship between the total cost and the number of items can be expressed as $t = pn$.</i> Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0, 0)$ and $(1, r)$ where r is the unit rate.
	M.7.RP.A.3	Use proportional relationships to solve multi-step ratio and percent problems. <i>Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.</i>

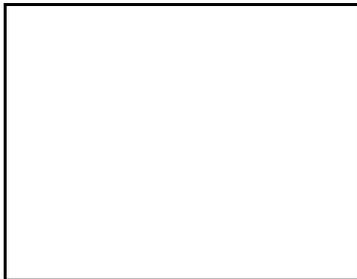
The Number System (7.NS)

Cluster Statement	Notation	Standard
<p>A. Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.</p>	M.7.NS.A.1	<p>Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.</p> <p>a. Describe situations in which opposite quantities combine to make 0. Show that a number and its opposite have a sum of 0 (are additive inverses).</p> <p><i>For example, if you earn \$10 and then spend \$10, you are left with \$0.</i></p> <p>b. Understand $p + q$ as the number located a distance q from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.</p> <p>c. Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.</p> <p>d. Apply properties of operations as strategies to add and subtract rational numbers.</p>
	M.7.NS.A.2	<p>Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.</p> <p>a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.</p>

		<ul style="list-style-type: none"> b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(-q)$. Interpret quotients of rational numbers by describing real-world contexts. c. Apply properties of operations as strategies to multiply and divide rational numbers. d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.
	M.7.NS.A.3	<p>Solve real-world and mathematical problems involving the four operations with rational numbers.</p> <p>Computations with rational numbers extend the rules for manipulating fractions to complex fractions.</p>

The Expressions and Equations (7.EE)

Cluster Statement	Notation	Standard
A. Use properties of operations to generate equivalent expressions.	M.7.EE.A.1	Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients.
	M.7.EE.A.2	Understand that rewriting an expression in different forms in a problem context can shed light on the problem and how the quantities in it are related. <i>For example, $a + 0.05a = 1.05a$ means that "increase by 5%" is the same as "multiply by 1.05."</i>
B. Solve real-life and mathematical problems using numerical and algebraic expressions and equations.	M.7.EE.B.3	Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. <i>For example: If a woman making \$25 an hour gets a 10% raise, she will make an additional 1/10 of her salary an hour, or \$2.50, for a new salary of \$27.50.</i>
	M.7.EE.B.4	Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. a. Solve word problems leading to equations of the form $px + q = r$ and $p(x + q) = r$, where p , q , and r are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. <i>For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?</i>



- b. Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p , q , and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

For example: As a salesperson, you are paid \$50 per week plus \$3 per sale. This week you want your pay to be at least \$100. Write an inequality for the number of sales you need to make, and describe the solutions.

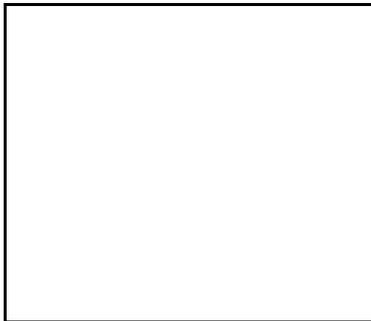
Geometry (7.G)

Cluster Statement	Notation	Standard
A. Draw, construct, and describe geometrical figures and describe the relationships between them.	M.7.G.A.1	Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
	M.7.G.A.2	Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
	M.7.G.A.3	Describe the two-dimensional figures that result from slicing three dimensional figures <i>parallel to the base</i> , as in plane sections of right rectangular prisms and right rectangular pyramids.
B. Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.	M.7.G.B.4	Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.
	M.7.G.B.5	Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.
	M.7.G.B.6	Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Statistics and Probability (7.SP)

Cluster Statement	Notation	Standard
A. Use random sampling to draw inferences about a population.	M.7.SP.A.1	Understand that statistics can be used to gain information about a population by examining a sample of the population; generalizations about a population from a sample are valid only if the sample is representative of that population. Understand that random sampling tends to produce representative samples and support valid inferences.
	M.7.SP.A.2	Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions. <i>For example, estimate the mean word length in a book by randomly sampling words from the book; predict the winner of a school election based on randomly sampled survey data. Gauge how far off the estimate or prediction might be.</i>
B. Draw informal comparative inferences about two populations.	M.7.SP.B.3	Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability. <i>For example, the mean height of players on the basketball team is 10 cm greater than the mean height of players on the soccer team, about twice the variability (mean absolute deviation) on either team; on a dot plot, the separation between the two distributions of heights is noticeable.</i>
	M.7.SP.B.4	Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations. <i>For example, decide whether the words in a chapter of a seventh-grade science book are generally longer than the words in a chapter of a fourth-grade science book.</i>

C. Investigate chance processes and develop, use, and evaluate probability models.	M.7.SP.C.5	Understand that the probability of a chance event is a number between 0 and 1 that expresses the likelihood of the event occurring. Larger numbers indicate greater likelihood. A probability near 0 indicates an unlikely event, a probability around $\frac{1}{2}$ indicates an event that is neither unlikely nor likely, and a probability near 1 indicates a likely event.
	M.7.SP.C.6	Approximate the probability of a chance event by collecting data on the chance process that produces it and observing its long-run relative frequency, and predict the approximate relative frequency given the probability. <i>For example, when rolling a number cube 600 times, predict that a 3 or 6 would be rolled roughly 200 times, but probably not exactly 200 times.</i>
	M.7.SP.C.7	Develop a probability model and use it to find probabilities of events. Compare probabilities from a model to observed frequencies; if the agreement is not good, explain possible sources of the discrepancy. <ul style="list-style-type: none"> a. Develop a uniform probability model by assigning equal probability to all outcomes, and use the model to determine probabilities of events. <i>For example, if a student is selected at random from a class, find the probability that Jane will be selected and the probability that a girl will be selected.</i> b. Develop a probability model (which may not be uniform) by observing frequencies in data generated from a chance process. <i>For example, find the approximate probability that a spinning penny will land heads up or that a tossed paper cup will land open-end down. Do the outcomes for the spinning penny appear to be equally likely based on the observed frequencies?</i>
	M.7.SP.C.8	Find probabilities of compound events using organized lists, tables, tree diagrams, and simulation. <ul style="list-style-type: none"> a. Understand that, just as with simple events, the probability of a compound event is the fraction of outcomes in the sample space for which the compound event occurs.



- b. Represent sample spaces for compound events using methods such as organized lists, tables and tree diagrams. For an event described in everyday language (e.g., "rolling double sixes"), identify the outcomes in the sample space which compose the event.
- c. Design and use a simulation to generate frequencies for compound events.
For example, use random digits as a simulation tool to approximate the answer to the question: If 40% of donors have type A blood, what is the probability that it will take at least 4 donors to find one with type A blood?

Introduction: Grade 8

In Grade 8, instructional time should focus on three critical areas: formulating and reasoning about expressions and equations, including modeling an association in bivariate data with a linear equation, and solving linear equations and systems of linear equations; grasping the concept of a function and using functions to describe quantitative relationships; analyzing two- and three-dimensional space and figures using distance, angle, similarity, and congruence, and understanding and applying the Pythagorean Theorem. Not all content in a given grade is emphasized equally in the Standards. Some concepts/skills require greater emphasis than others based on the depth of the ideas, the time that they take to master, and/or their importance to future mathematics or the demands of college and career readiness. More time in these areas is also necessary for students to meet the Standards for Mathematical Practice. To say that some things have greater emphasis is not to say that anything in the Standards can safely be neglected in instruction. Neglecting material will leave gaps in student skill and understanding and may leave students unprepared for the challenges of a later grade (Student Achievement Partners 2014).

Students use linear equations and systems of linear equations to represent, analyze, and solve a variety of problems. Students recognize equations for proportions ($y/x = m$ or $y = mx$) as special linear equations ($y = mx + b$), understanding that the constant of proportionality (m) is the slope, and the graphs are lines through the origin. They understand that the slope (m) of a line is a constant rate of change, so that if the input or x -coordinate changes by an amount A , the output or y -coordinate changes by the amount $m \cdot A$. Students also use a linear equation to describe the association between two quantities in bivariate data (such as arm span vs. height for students in a classroom). At this grade, fitting the model, and assessing its fit to the data are done informally. Interpreting the model in the context of the data requires students to express a relationship between the two quantities in question and to interpret components of the relationship (such as slope and y -intercept) in terms of the situation. Students strategically choose and efficiently implement procedures to solve linear equations in one variable, understanding that when they use the properties of equality and the concept of logical equivalence, they maintain the solutions of the original equation. Students solve systems of two linear equations in two variables and relate the systems to pairs of lines in the plane; these intersect, are parallel, or are the same line. Students use linear equations, systems of linear equations, linear functions, and their understanding of slope of a line to analyze situations and solve problems.

Students grasp the concept of a function as a rule that assigns to each input exactly one output. They understand that functions describe situations where one quantity determines another. They can translate among representations and partial

representations of functions (noting that tabular and graphical representations may be partial representations), and they describe how aspects of the function are reflected in the different representations.

Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. They apply the Pythagorean Theorem to find distances between points on the coordinate plane, to find lengths, and to analyze polygons. Students complete their work on volume by solving problems involving cones, cylinders, and spheres.

One way to empower students in becoming flexible users of mathematics is to provide authentic opportunities to engage in mathematical thinking. Mathematical modeling is a powerful way to engage students. At the middle school level, the big idea within mathematical modeling is encouraging students to act more independently as they attack bigger problems with their growing collection of mathematical tools. This may include considering a big question as a whole class while small groups of students look at a particular aspect of the problem that interest them, making more sophisticated arguments and taking into consideration several constraints at the same time, students critiquing their own work as they report their results, and demonstrating and discussing what happens to their solution when they change an assumption or a particular number (GAIMME 2016, 34).

Grade 8 Overview

The Number System

- Know that there are numbers that are not rational, and approximate them by rational numbers.

Expressions and Equations

- Work with radicals and integer exponents.
- Understand the connections between proportional relationships, lines, and linear equations.
- Analyze and solve linear equations and pairs of simultaneous linear equations.

Functions

- Define, evaluate, and compare functions.
- Use functions to model relationships between quantities.

Geometry

- Understand congruence and similarity using physical models, transparencies, or geometry software.
- Understand and apply the Pythagorean Theorem.
- Solve real-world and mathematical problems involving volume of cylinders, cones and spheres.

Statistics and Probability

- Investigate patterns of association in bivariate data.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Grade 8 Content Standards

The Number System (8.NS)

Cluster Statement	Notation	Standard
A. Know that there are numbers that are not rational, and approximate them by rational numbers.	M.8.NS.A.1	Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and use patterns to rewrite a decimal expansion that repeats into a rational number.
	M.8.NS.A.2	Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). <i>For example, by truncating the decimal expansion of $\sqrt{2}$, show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.</i>

The Expressions and Equations (8.EE)

Cluster Statement	Notation	Standard
A. Work with radicals and integer exponents.	M.8.EE.A.1	Know and apply the properties of integer exponents to generate equivalent numerical expressions. <i>For example, $3^2 \times 3^{-5} = 3^{-3} = 1/3^3 = 1/27$.</i>
	M.8.EE.A.2	Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
	M.8.EE.A.3	Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. <i>For example, estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.</i>
	M.8.EE.A.4	Perform operations with numbers expressed in scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology and comparing magnitude of numbers.
B. Understand the connections between proportional	M.8.EE.B.5	Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways. <i>For example, compare a distance-time graph to a distance-time equation to determine which of two moving objects has greater speed.</i>

relationships, lines, and linear equations.	M.8.EE.B.6	Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b .
C. Analyze and solve linear equations and pairs of simultaneous linear equations.	M.8.EE.C.7	<p>Solve linear equations in one variable.</p> <ol style="list-style-type: none"> Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into equivalent forms. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.
	M.8.EE.C.8	<p>Analyze and solve pairs of simultaneous linear equations.</p> <ol style="list-style-type: none"> Understand that solutions to a system of two linear equations in two variables correspond to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously. Solve systems of two linear equations in two variables by graphing and analyzing tables. Solve simple cases represented in algebraic symbols by inspection. <i>For example, $3x + 2y = 5$ and $3x + 2y = 6$ cannot simultaneously be 5 and 6.</i> Solve real-world and mathematical problems leading to two linear equations in two variables. <i>For example, given coordinates for two pairs of points, determine whether the line through the first pair of points intersects the line through the second pair.</i>

Functions (8.F)

Cluster Statement	Notation	Standard
A. Define, evaluate, and compare functions.	M.8.F.A.1	Understand that a function is a rule that assigns to each input exactly one output. The graph of a numerically valued function is the set of ordered pairs consisting of an input and the corresponding output. (Function notation is not required in Grade 8.)
	M.8.F.A.2	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). <i>For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.</i>
	M.8.F.A.3	Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. <i>For example, the function $A = s^2$ giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.</i>
B. Use functions to model relationships between quantities.	M.8.F.B.4	Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
	M.8.F.B.5	Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear, continuous or discrete). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Geometry (8.G)

Cluster Statement	Notation	Standard
A. Understand congruence and similarity using physical models, transparencies, or geometry software.	M.8.G.A.1	Verify experimentally the properties of rotations, reflections, and translations: <ol style="list-style-type: none"> a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines.
	M.8.G.A.2	Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
	M.8.G.A.3	Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
	M.8.G.A.4	Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
	M.8.G.A.5	Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. <i>For example, arrange three copies of the same triangle so that the sum of the three angles appears to form a line, and give an argument in terms of transversals why this is so.</i>

B. Understand and apply the Pythagorean Theorem.	M.8.G.B.6	Justify the relationship between the lengths of the legs and the length of the hypotenuse of a right triangle, and the converse of the Pythagorean theorem.
	M.8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	M.8.G.B.8	Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
C. Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.	M.8.G.C.9	Know the relationship among the formulas for the volumes of cones, cylinders, and spheres (given the same height and diameter) and use them to solve real-world and mathematical problems.

Statistics and Probability (8.SP)

Cluster Statement	Notation	Standard
A. Investigate patterns of association in bivariate data.	M.8.SP.A.1	Construct and interpret scatter plots for bivariate measurement data to investigate patterns of association between two quantities. Describe patterns such as clustering, outliers, positive or negative association, linear association, and nonlinear association.
	M.8.SP.A.2	Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.
	M.8.SP.A.3	Use the equation of a linear model to solve problems in the context of bivariate measurement data, interpreting the slope and intercept. <i>For example, in a linear model for a biology experiment, interpret a slope of 1.5 cm/hr as meaning that an additional hour of sunlight each day is associated with an additional 1.5 cm in mature plant height.</i>
	M.8.SP.A.4	Understand that patterns of association can also be seen in bivariate categorical data by displaying frequencies and relative frequencies in a two-way table. Construct and interpret a two-way table summarizing data on two categorical variables collected from the same subjects. Use relative frequencies calculated for rows or columns to describe possible association between the two variables. <i>For example, collect data from students in your class on whether or not they have a curfew on school nights and whether or not they have assigned chores at home. Is there evidence that those who have a curfew also tend to have chores?</i>

High School Standards

When the multiple purposes of school mathematics are continued to be emphasized in high school mathematics, students are prepared to “flourish as human beings” in whatever path or profession they choose (Su 2017, p.483). The high school standards work to both continue developing students as mathematical thinkers building on the students’ mathematical development and empowerment in Grades 6-8 and preparing students for a variety of mathematical opportunities beyond high school graduation.

The standards support the transition from high school to post-secondary education for college and careers. Using the standards to support various pathways in mathematics (e.g., STEM-related courses, Statistics (or Data Science), and Applied Mathematics to name a few) can support equitable access to mathematics to support the transition to post-secondary options. The standards are presented here in such a way to allow for districts to be flexible in course offerings in grades 9-12. The high school portion of the *Wisconsin Standards for Mathematics (2021)* specifies the mathematics all students should study for college and career and life readiness. These standards do not mandate the sequence of high school courses. However, the organization of high school courses is a critical component to implementation of the standards. To that end, high school courses tend to follow one of two pathways for mathematics, 1) a traditional course sequence (Algebra I, Geometry, and Algebra II) or 2) an integrated course sequence (Mathematics 1, Mathematics 2, and Mathematics 3). There are increasingly more options for pathways given the call for access and equity and for the variety of the options for post-secondary options. More information is available in the appendix and on the DPI website.

The high school standards are listed in conceptual categories:

- Number and Quantity
- Algebra
- Functions
- Modeling
- Geometry
- Statistics and Probability

Conceptual categories portray a coherent view of high school mathematics; a student’s work with functions, for example, crosses a number of traditional course boundaries, potentially up through and including calculus. Mathematical modeling is best completed and interpreted not as a collection of isolated topics but integrated into other content standards. Making mathematical models is the Fourth Standard for Mathematical Practice, and specific modeling standards appear throughout the high school content standards indicated by an (M) symbol. The (M) symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

All standards without a (+) symbol should be in the common mathematics curriculum for all students to support them in their choices in post-secondary options. Standards with a (+) symbol may also appear in courses intended for all students. Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics are indicated by (+), as in this example:

(+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers).

Access and equity in mathematics at the school and classroom levels is founded on beliefs and practices that empower each and every student to participate meaningfully in learning mathematics and to achieve outcomes in mathematics that are not predicted by or associated with student characteristics. For all students, mathematics is an intellectually challenging activity that transcends their racial, ethnic, linguistic, gender, and socioeconomic backgrounds. Promoting curiosity and wonder through mathematical discourse is possible when schools and classrooms provide equitable access to challenging curriculum and set high expectations for all students.⁴ The standards alone will not be sufficient in considerations for course work. *School leaders, teachers, and community stakeholders need to collaborate on issues impacting access and equity for each and every student, such as tracking, beliefs about innate levels of mathematical ability, and differentiated learning. To gain more insight, read and discuss “Access and Equity” in Principles to Actions (pp. 59-69).*

High School Standards for Mathematical Practice

Math Practice 1: Make sense of problems and persevere in solving them.

HS Mathematically proficient high school students analyze givens, constraints, relationships, and goals. They make assumptions where needed to make the problem more clearly articulated. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. Students identify questions to ask and make observations about the situation through notice and wondering strategies. While following a solution plan, they continually ask themselves, “Does this make sense?” They monitor and evaluate their progress and revise their plan as needed. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. High school students might, depending on the context of the problem, transform algebraic expressions to provide them with different information about the situation. They change the display on their graphing tool to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph and interpret representations of data, and search for regularity or trends. Mathematically proficient students gain deeper insight into problems by using a different approach, understanding the approaches of others to solving complex problems, and identifying correspondences between different approaches. Mathematically proficient students are engaged in the problem-solving process, do not give up when stuck, and accept that it is acceptable to proceed forward when confronted with confusion and struggle.

Math Practice 2: Reason abstractly and quantitatively.

HS Mathematically proficient high school students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. For example, high school students might work with an exponential function given in symbolic form, but represents a contextual situation. Students are able to manipulate and change the form of the function and reveal different information about the situation based on the numbers stated in the algebraic representation. In addition, students can write explanatory text that conveys their mathematical analyses and thinking, using relevant and sufficient facts, concrete details, quotations, and coherent discussion of ideas. Students can evaluate multiple sources of information presented in diverse formats (and media) to address a question or solve a problem. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Math Practice 3: Construct viable arguments, and appreciate and critique the reasoning of others. (Gutierrez 2017)

HS Mathematically proficient high school students understand and use stated assumptions, definitions, and previously established results in constructing verbal and written arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples and specific textual evidence to form their arguments. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is and why. They can construct formal arguments relevant to specific contexts and tasks. High school students learn to determine domains to which an argument applies. Students listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. Students engage in collaborative discussions, respond thoughtfully to diverse perspectives and approaches, and qualify their own views in light of evidence presented. They can present their findings and results to a given audience through a variety of formats such as posters, whiteboards, and interactive materials.

Math Practice 4: Model with mathematics.

HS Mathematically proficient high school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. Students that engage in modeling have choice when solving problems. By high school, a student might use geometry to solve a design problem or build a function to describe how one quantity depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts, and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. They can carry out all phases of the modeling cycle as outlined in the shift for mathematical modeling (reference). Mathematically proficient high school students also retain the widely applicable techniques they first learned in middle school, such as proportional relationships, rates, and percentages, and apply these techniques as needed to real-world tasks of a complexity appropriate to high school.

Note: Although physical objects and visuals can be used to model a situation, using these tools absent a contextual situation is not an example of practice standard MP.4. For example, using algebra tiles or an area model to illustrate factoring a quadratic expression would not be an example of practice standard MP.4. Practice standard MP.4 is about applying math to a problem in context.

Math Practice 5: Use appropriate tools strategically.

HS Mathematically proficient high school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for high school to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students can use slider bars in a dynamic calculator in order to what-if a situation and see how the graph of a function changes when the parameters in the equation are changed. They can use a spreadsheet to model change when cells are dynamically linked together and values are changed. Students can analyze graphs of functions and solutions generated using a graphing calculator; they also know how to sketch graphs of common functions, choosing this approach over a graphing calculator when a sketch will suffice (MP.6). They detect possible errors by strategically using estimation and other mathematical knowledge, for example anticipating the general appearance of a graph of a function by identifying the structure of its defining expression (MP.7). They are able to use software or websites to quickly generate data displays that would otherwise be time-consuming to construct by hand (such as histograms, box plots, or simulation models for random sampling). Students use technological tools to explore and deepen their understanding of mathematical concepts and analyze realistic data sets. When making mathematical models, students know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. (MP.4) Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

Math Practice 6: Attend to precision.

HS Mathematically proficient high school students communicate precisely to others both verbally and in writing, adapting their communication to specific contexts, audiences, and purposes. They increasingly use precise language, not only as a mechanism for effective communication, but also as a tool for understanding and solving problems. Describing their ideas precisely helps students understand the ideas in new ways. They use clear definitions in discussions with others and in their own reasoning. They state the meaning of the symbols that they choose. They are careful about specifying units of measure, labeling axes, defining terms and variables, and calculating accurately and efficiently with a degree of precision appropriate for the problem context. They present logical claims and counterclaims fairly and thoroughly in a way that anticipates the audiences' knowledge, concerns, and possible biases. High school students draw specific evidence from informational sources to support analysis, reflection, and research. They critically evaluate the claims, evidence and reasoning of others and attend to important distinctions with their own claims or inconsistencies in competing claims. Students evaluate the conjectures and claims, data, analysis, and conclusions in texts that include quantitative elements, comparing those with information found in other sources. Diligence and attention to detail are mathematical virtues: mathematically proficient students care that an answer is right; they minimize

errors by keeping a long calculation organized; they check their work; they solve the problem another way; they take responsibility for careless mistakes and correct them.

Math Practice 7: Look for and make use of structure.

HS Mathematically proficient high school students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, high school students can see the 14 as 2×7 and the 9 as $2 + 7$. In an equation, high school students recognize that $12 = 3(x-1)^2$ does not require distribution in the expression on the right in order to carry out the process of solving for x . They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers x and y . Students make use of structure for a purpose, for example by applying the conclusion $5 - 3(x - y)^2 \leq 5$ in the context of an applied optimization problem. Students will notice that the structure of a quadratic function written symbolically in a - b - c form (standard), vertex form, or factored form will reveal different information about the graph of the function.

Math Practice 8: Look for and express regularity in repeated reasoning.

HS Mathematically proficient high school students notice if calculations are repeated, and look both for general and efficient methods. Noticing the regularity in the way terms sum to zero when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead students to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details and continually evaluating the reasonableness of their intermediate results. When students repeatedly compute products of the form $(ax + b)(ax + b)$ and notice the pattern equals $(a^2x^2 + 2abx + b^2)$ they are looking for and expressing regularity in repeated reasoning. Students change their perspective or view and use what they know. They turn or break down structure to something they know. They solve tasks by solving a sub-problem or smaller version. They might add a line or turn geometric structures, so they identify something they have worked with before. By doing this it helps students move forward and not be stuck. Students use repetition in reasoning as they work with various expressions involving exponents and develop an understanding of the various structures. Students connect Pascal's triangle to the repetition in reasoning that occurs in the expansion of binomial coefficients.

Mathematics - High School Number & Quantity: Introduction

Numbers and Number Systems

During the years from kindergarten to eighth grade, students must repeatedly extend their conception of the term number. At first, “number” means “counting number”: 1, 2, 3, ... Soon after that, 0 is used to represent “none” and the whole numbers are formed by the counting numbers together with zero. The next extension is fractions. At first, fractions are barely numbers and tied strongly to pictorial representations. Yet by the time students understand division of fractions, they have a strong concept of fractions as numbers and have connected them, via their decimal representations, with the base-ten system used to represent the whole numbers. During middle school, fractions are augmented by negative fractions to form the rational numbers. In Grade 8, students extend this system once more, augmenting the rational numbers with the irrational numbers to form the real numbers. In high school, students will be exposed to yet another extension of number, when the real numbers are augmented by the imaginary numbers to form the complex numbers.

With each extension of number, the meanings of addition, subtraction, multiplication, and division are extended. In each new number system—integers, rational numbers, real numbers, and complex numbers—the four operations stay the same in two important ways: They have the commutative, associative, and distributive properties and their new meanings are consistent with their previous meanings.

Extending the properties of whole-number exponents leads to new and productive notation. For example, properties of whole-number exponents suggest that $(5^{1/3})^3$ should be $5^{(1/3)3} = 5^1 = 5$ and that $5^{1/3}$ should be the cube root of 5.

Calculators, spreadsheets, and computer algebra systems can provide ways for students to become better acquainted with these new number systems and their notation. Students have the opportunity to generate data for numerical experiments, to help understand the workings of matrix, vector, and complex number algebra, and to experiment with non-integer exponents.

Quantities

In real-world problems, the answers are usually not numbers but quantities: numbers with units, which involves measurement. In their work in measurement up through Grade 8, students primarily measure commonly used attributes such as length, area, and volume. In high school, students encounter a wider variety of units in modeling (e.g., acceleration, currency conversions, derived quantities such as person-hours and heating degree days, social science rates such as per-capita income, and rates in everyday life such as points scored per game or batting averages). They also encounter novel situations in which they themselves must

conceive the attributes of interest. For example, to find a good measure of overall highway safety, they might propose measures such as fatalities per year, fatalities per year per driver, or fatalities per vehicle-mile traveled. Such a conceptual process might be called quantification. Quantification is important for science, as when surface area suddenly “stands out” as an important variable in evaporation. Quantification is also important for companies, which must conceptualize relevant attributes and create or choose suitable measures for them.

Number and Quantity Overview

The Real Number System

- Extend the properties of exponents to rational exponents.
- Use properties of rational and irrational numbers.

Quantities

- Reason quantitatively and use units to solve problems.

The Complex Number System

- Perform arithmetic operations with complex numbers.
- Represent complex numbers and their operations on the complex plane.
- Use complex numbers in polynomial identities and equations.

Vector and Matrix Quantities

- Represent and model with vector quantities.
- Perform operations on vectors.
- Perform operations on matrices and use matrices in applications.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

The Real Number System (N-RN)

Cluster Statement	Notation	Standard
A. Extend the properties of exponents to rational exponents.	M.N.RN.A.1	Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents.
	M.N.RN.A.2	Rewrite expressions involving radicals and rational exponents using the properties of exponents.
B. Use properties of rational and irrational numbers.	M.N.RN.A.3	Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Quantities (N-Q)

Cluster Statement	Notation	Standard
A. Reason quantitatively and use units to solve problems.	M.N.Q.A.1	Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays. (M)
	M.N.Q.A.2	Define appropriate quantities for the purpose of descriptive modeling. (M)
	M.N.Q.A.3	Choose a level of accuracy appropriate to limitations on measurement when reporting quantities. (M)

The Complex Number System (N-CN)

Cluster Statement	Notation	Standard
A. Perform arithmetic operations with complex numbers.	M.N.CN.A.1	Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.
	M.N.CN.A.2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
	M.N.CN.A.3	Find the conjugate of a complex number; use conjugates to find moduli (absolute values) and quotients of complex numbers.
B. Represent complex numbers and their operations on the complex plane.	M.N.CN.A.4	Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.
	M.N.CN.A.5	Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. <i>For example, $(-1 + \sqrt{3}i)^3 = 8$ because $(-1 + \sqrt{3}i)$ has modulus 2 and argument 120°.</i>
	M.N.CN.A.6	(+) Calculate the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints.
C. Use complex numbers in	M.N.CN.A.7	Solve quadratic equations with real coefficients that have complex solutions.

polynomial identities and equations.	M.N.CN.A.8	(+) Extend polynomial identities to the complex numbers. <i>For example, rewrite $x^2 + 4$ as $(x + 2i)(x - 2i)$.</i>
	M.N.CN.A.9	(+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

Vector and Matrix Quantities (N-VM)

Cluster Statement	Notation	Standard
A. Represent and model with vector quantities.	M.N.VM.A.1	(+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g., \mathbf{v} , $ \mathbf{v} $, $\ \mathbf{v}\ $, v).
	M.N.VM.A.2	(+) Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
	M.N.VM.A.3	(+) Solve problems involving velocity and other quantities that can be represented by vectors.
B. Perform operations on vectors.	M.N.VM.B.4	<p>(+) Add and subtract vectors.</p> <p>a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.</p> <p>b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.</p> <p>c. Understand vector subtraction $\mathbf{v} - \mathbf{w}$ as $\mathbf{v} + (-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of \mathbf{w}, with the same magnitude as \mathbf{w} and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.</p>
	M.N.VM.B.5	(+) Multiply a vector by a scalar.

		<p>a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c(v_x, v_y) = (cv_x, cv_y)$.</p> <p>b. Compute the magnitude of a scalar multiple cv using $\ cv\ = c v$. Compute the direction of cv knowing that when $c v \neq 0$, the direction of cv is either along v (for $c > 0$) or against v (for $c < 0$).</p>
C. Perform operations on matrices and use matrices in applications.	M.N.VM.C.6	(+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.
	M.N.VM.C.7	(+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
	M.N.VM.C.8	(+) Add, subtract, and multiply matrices of appropriate dimensions.
	M.N.VM.C.9	(+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.
	M.N.VM.C.10	(+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
	M.N.VM.C.11	(+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.



M.N.VM.C.12

(+) Work with 2×2 matrices as transformations of the plane, and interpret the absolute value of the determinant in terms of area.

Mathematics - High School Algebra: Introduction

Expressions

An expression is a record of a computation with numbers, symbols that represent numbers, arithmetic operations, exponentiation, and, at more advanced levels, the operation of evaluating a function. Conventions about the use of parentheses and the order of operations assure that each expression is unambiguous. Creating an expression that describes a computation involving a general quantity requires the ability to express the computation in general terms, abstracting from specific instances.

Reading an expression with comprehension involves analysis of its underlying structure. This may suggest a different but equivalent way of writing the expression that exhibits some different aspect of its meaning. For example, $p + 0.05p$ can be interpreted as the addition of a 5% tax to a price p . Rewriting $p + 0.05p$ as $1.05p$ shows that adding a tax is the same as multiplying the price by a constant factor.

Algebraic manipulations are governed by the properties of operations and exponents, and the conventions of algebraic notation. At times, an expression is the result of applying operations to simpler expressions. For example, $p + 0.05p$ is the sum of the simpler expressions p and $0.05p$. Viewing an expression as the result of operation on simpler expressions can sometimes clarify its underlying structure.

A spreadsheet or a computer algebra system (CAS) can be used to experiment with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave.

Equations and inequalities

An equation is a statement of equality between two expressions, often viewed as a question asking for which values of the variables the expressions on either side are in fact equal. These values are the solutions to the equation. An identity, in contrast, is true for all values of the variables; identities are often developed by rewriting an expression in an equivalent form.

The solutions of an equation in one variable form a set of numbers; the solutions of an equation in two variables form a set of ordered pairs of numbers, which can be plotted in the coordinate plane. Two or more equations and/or inequalities form a system. A solution for such a system must satisfy every equation and inequality in the system.

An equation can often be solved by successively deducing from it one or more simpler equations. For example, one can add the same constant to both sides without changing the solutions, but squaring both sides might lead to extraneous solutions. Strategic competence in solving includes looking ahead for productive manipulations and anticipating the nature and number of solutions.

Some equations have no solutions in a given number system, but have a solution in a larger system. For example, the solution of $x + 1 = 0$ is an integer, not a whole number; the solution of $2x + 1 = 0$ is a rational number, not an integer; the solutions of $x^2 - 2 = 0$ are real numbers, not rational numbers; and the solutions of $x^2 + 2 = 0$ are complex numbers, not real numbers.

The same solution techniques used to solve equations can be used to rearrange formulas. For example, the formula for the area of a trapezoid, $A = ((b_1 + b_2)/2)h$, can be solved for h using the same deductive process.

Inequalities can be solved by reasoning about the properties of inequality. Many, but not all, of the properties of equality continue to hold for inequalities and can be useful in solving them.

Algebra Overview

Seeing Structure in Expressions

- Interpret the structure of expressions.
- Write expressions in equivalent forms to solve problems.

Arithmetic with Polynomials and Rational Expressions

- Perform arithmetic operations on polynomials.
- Understand the relationship between zeros and factors of polynomials.
- Use polynomial identities to solve problems.
- Rewrite rational expressions.

Creating Equations

- Create equations that describe numbers or relationships.

Reasoning with Equations and Inequalities

- Understand solving equations as a process of reasoning and explain the reasoning.
- Solve equations and inequalities in one variable.
- Solve systems of equations.
- Represent and solve equations and inequalities graphically.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Algebra Content Standards

Seeing Structure in Expressions (A-SSE)

Cluster Statement	Notation	Standard
A. Interpret the structure of expressions.	M.A.SSE.A.1	<p>Interpret expressions that represent a quantity in terms of its context. (M)</p> <ul style="list-style-type: none"> a. Interpret parts of an expression, such as terms, factors, and coefficients. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. <p><i>For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.</i></p>
	M.A.SSE.A.2	<p>Use the structure of an expression to identify ways to rewrite it.</p> <p><i>For example, see x^4-y^4 as $(x^2)^2-(y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2-y^2)(x^2+y^2)$.</i></p>
B. Write expressions in equivalent forms to solve problems.	M.A.SSE.B.3	<p>Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.</p> <ul style="list-style-type: none"> a. Factor a quadratic expression to reveal the zeros of the function it defines. b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions.

		<p><i>For example, the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.</i></p>
	<p>M.A.SSE.B.4</p>	<p>Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. (M)</p> <p><i>For example, calculate mortgage payments.</i></p>

Arithmetic with Polynomials and Rational Expressions (A-APR)

Cluster Statement	Notation	Standard
A. Perform arithmetic operations on polynomials.	M.A.APR.A.1	Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.
B. Understand the relationship between zeros and factors of polynomials.	M.A.APR.B.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a , the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.
	M.A.APR.B.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
C. Use polynomial identities to solve problems.	M.A.APR.C.4	Prove polynomial identities and use them to describe numerical relationships. <i>For example, the polynomial identity $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ can be used to generate Pythagorean triples.</i>
	M.A.APR.C.5	(+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n , where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.
	M.A.APR.A.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$,

D. Rewrite rational expressions.		using inspection, long division, or, for the more complicated examples, a computer algebra system.
	M.A.APR.A.7	(+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Creating Equations (A-CED)

Cluster Statement	Notation	Standard
A. Create equations that describe numbers or relationships.	M.A.CED.A.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. (M)
	M.A.CED.A.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. (M)
	M.A.CED.A.3	Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. (M) <i>For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.</i>
	M.A.CED.A.4	Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. <i>For example, rearrange the formula for the area of a trapezoid to highlight one of the bases.</i>

Reasoning with Equations and Inequalities (A-REI)

Cluster Statement	Notation	Standard
A. Understand solving equations as a process of reasoning and explain the reasoning.	M.A.REI.A.1	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	M.A.REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
B. Solve equations and inequalities in one variable.	M.A.REI.B.3	Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
	M.A.REI.B.4	Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula, factoring, and graphing as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b .
	M.A.REI.C.5	Justify/Verify that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

C. Solve systems of equations.	M.A.REI.C.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	M.A.REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. <i>For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.</i>
	M.A.REI.C.8	(+) Represent a system of linear equations as a single matrix equation in a vector variable.
	M.A.REI.C.9	(+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3×3 or greater).
D. Represent and solve equations and inequalities graphically.	M.A.REI.D.10	Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
	M.A.REI.D.11	Explain why the <i>x-coordinates</i> of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.
	M.A.REI.D.12	Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Mathematics - High School Functions: Introduction

Functions describe situations where one quantity determines another. For example, the return on \$10,000 invested at an annualized percentage rate of 4.25% is a function of the length of time the money is invested. Because we continually make theories about dependencies between quantities in nature and society, functions are important tools in the construction of mathematical models.

In school mathematics, functions usually have numerical inputs and outputs and are often defined by an algebraic expression. For example, the time in hours it takes for a car to drive 100 miles is a function of the car's speed in miles per hour, v ; the rule $T(v) = 100/v$ expresses this relationship algebraically and defines a function whose name is T .

The set of inputs to a function is called its domain. We often infer the domain to be all inputs for which the expression defining a function has a value, or for which the function makes sense in a given context.

A function can be described in various ways, such as by a graph (e.g., the trace of a seismograph); by a verbal rule, as in, "I'll give you a state, you give me the capital city;" by an algebraic expression like $f(x) = a + bx$; or by a recursive rule. The graph of a function is often a useful way of visualizing the relationship of the function models, and manipulating a mathematical expression for a function can throw light on the function's properties.

Functions presented as expressions can model many important phenomena. Two important families of functions characterized by laws of growth are linear functions, which grow at a constant rate, and exponential functions, which grow at a constant percent rate. Linear functions with a constant term of zero describe proportional relationships.

A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions.

Connections to Expressions, Equations, Modeling, and Coordinates

Determining an output value for a particular input involves evaluating an expression; finding inputs that yield a given output involves solving an equation. Questions about when two functions have the same value for the same input lead to equations, whose solutions can be visualized from the intersection of their graphs. Because functions describe relationships between quantities, they are frequently used in modeling. Sometimes functions are defined by a recursive process, which can be displayed

effectively using a spreadsheet or other technology.

Function Overview

Interpreting Functions

- Understand the concept of a function and use function notation.
- Intercept functions that arise in applications in terms of the context.
- Analyze functions using different representations.

Building Functions

- Build a function that models a relationship between two quantities.
- Build new functions from existing functions.

Linear, Quadratic, and Exponential Models

- Construct and compare linear, quadratic, and exponential models and solve problems.
- Interpret expressions for functions in terms of the situation they model.

Trigonometric Functions

- Extend the domain of trigonometric functions using the unit circle.
- Model periodic phenomena with trigonometric functions.
- Prove and apply trigonometric identities.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Functions

Interpreting Functions (F-IF)

Cluster Statement	Notation	Standard
A. Understand the concept of a function and use function notation.	M.F.IF.A.1	Understand that a function from one set, discrete or continuous, (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range.
	M.F.IF.A.2	Use function notation, evaluate functions. and interpret statements that use function notation in terms of a context.
	M.F.IF.A.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. <i>For example, in an arithmetic sequence, $f(x) = f(x-1) + C$ or in a geometric sequence, $f(x) = f(x-1) * C$, where C is a constant.</i>
B. Interpret functions that arise in applications in terms of context.	M.F.IF.B.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. <i>Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity. (M)</i>
	M.F.IF.B.5	Relate the domain of a function to its graph and find an appropriate domain (discrete or continuous) in the context of the given problem.
	M.F.IF.B.6	Calculate and interpret the average rate of change of a linear or nonlinear function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

C. Analyze functions using different representations.	M.F.IF.C.7	<p>Graph functions expressed symbolically and show key features of the graph using an efficient method. (M)</p> <ol style="list-style-type: none"> a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions. c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. d. (+) Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available, and showing end behavior. e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.
	M.F.IF.C.8	<p>Write a function defined by an expression in equivalent forms to reveal and explain different properties of the function.</p> <ol style="list-style-type: none"> a. Use an efficient process to rewrite $f(x) = ax^2+bx+c$ as $f(x) = a(x-h)^2 + k$ or $f(x)=a(x-p)(x-q)$ to determine the characteristics of the function and interpret these in terms of a context. b. Use the properties of exponents to interpret expressions for exponential functions. <p><i>For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.</i></p>
	M.F.IF.C.9	<p>Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).</p>

		For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.
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Building Functions (F-BF)

Cluster Statement	Notation	Standard
A. Build a function that models a relationship between two quantities.	M.F.BF.A1	<p>Write a function that describes a relationship between two quantities. (M)</p> <ul style="list-style-type: none"> a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. <i>For example: Combining quad, linear, and constant to create a quadratic function.</i> c. Compose functions. <p><i>For example, if $T(y)$ is the temperature in the atmosphere as a function of height, and $h(t)$ is the height of a weather balloon as a function of time, then $T(h(t))$ is the temperature at the location of the weather balloon as a function of time.</i></p>
	M.F.BF.A.2	Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms. (M)
B. Build new functions from existing functions.	M.F.BF.B.3	Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k f(x)$, $f(kx)$, and $f(x + k)$ using transformations for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

	M.F.BF.B.4	<p>Find inverse functions.</p> <ol style="list-style-type: none"> Solve an equation of the form $f(x) = c$ for a function f that has an inverse and write an expression for the inverse. <i>For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.</i> (+) Verify by composition that one function is the inverse of another. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse. (+) Produce an invertible function from a non-invertible function by restricting the domain.
	M.F.BF.B5	<p>(+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.</p>

Linear, Quadratic, and Exponential Models (F-LE)

Cluster Statement	Notation	Standard
A. Construct and compare linear, quadratic, and exponential models and solve problems.	M.F.LE.A.1	Distinguish between situations that can be modeled with linear functions and with exponential functions. (M) <ul style="list-style-type: none"> a. Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals. b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another. c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.
	M.F.LE.A.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). (M)
	M.F.LE.A.3	Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function. (M)
	M.F.LE.A.4	For exponential models, express as a logarithm the solution to $abc^t = d$ where a , c , and d are numbers and the base b is 2, 10, or e ; evaluate the logarithm using technology. (M)
B. Interpret expressions for functions in terms of the situation they	M.F.LE.B.5	Interpret the parameters in a linear or exponential function in terms of a context. (M)

model.		
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Trigonometric Functions (F-TF)

Cluster	Notation	Standard
A. Extend the domain of the trigonometric functions of the unit circle.	M.F.TF.A.1	1. Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
	M.F.TF.A.2	2. Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
	M.F.TF.A.3	3. (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi/3$, $\pi/4$ and $\pi/6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi-x$, $\pi+x$, and $2\pi-x$ in terms of their values for x , where x is any real number.
	M.F.TF.A.4	4. (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.
B. Model periodic phenomena with trigonometric functions.	M.F.TF.B.5	5. Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline. (M)
	M.F.TF.B.6	6. (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.
	M.F.TF.B.7	7. (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context. (M)

C. Prove and apply trigonometric identities.	M.F.TF.C.8	8. Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
	M.F.TF.C.9	9. (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, formulate suggestions, and present them to an audience in order to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology may be valuable for varying assumptions, exploring consequences, and comparing predictions with data.

A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or if a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every such process, this depends on acquired expertise as well as creativity.

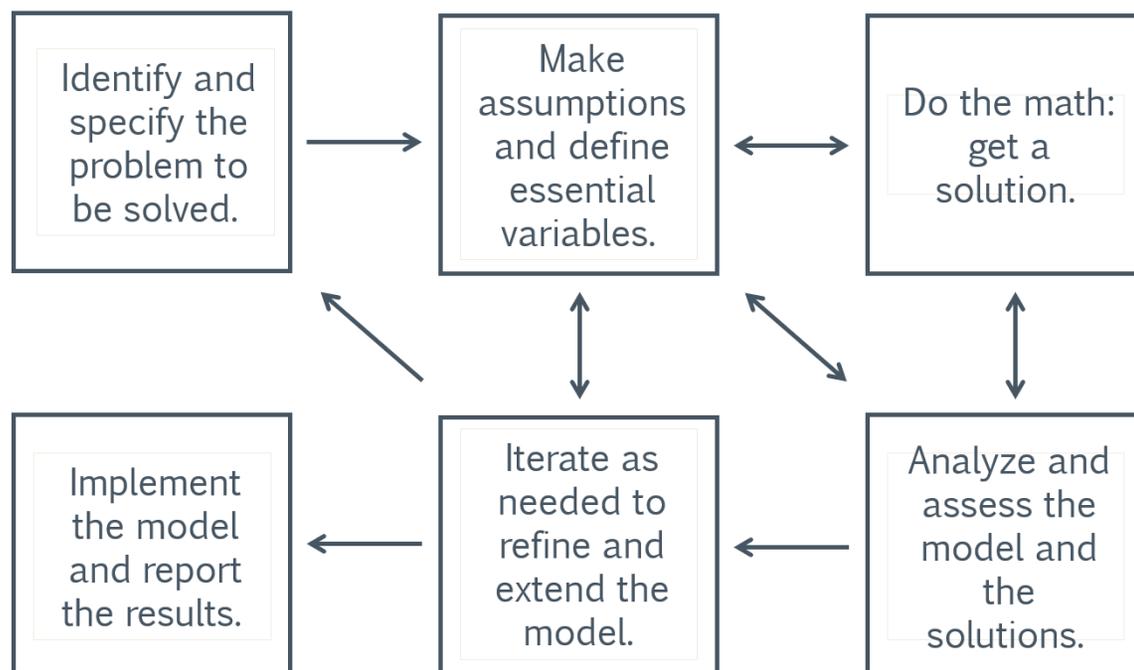
Some examples of such situations might include:

- Estimating a cell tower coverage when structural or electrical interference might play a significant factor.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the optimal location for a new school being built in a rural community.
- Analyzing cost savings in driving across town to a different gas station for a cheaper price per gallon of fuel.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Comparing cost of ownership for two different types of vehicles.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The

range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



Source: GAIMME Report, 2016

The basic modeling cycle is summarized in the diagram. (1) We identify something in the real-world we want to know, do, or understand. The result is a question in the real-world. (2) We select 'objects' that seem important in the real-world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original questions. (3) We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized question. This formulation is the model. We do the math to see what insights and results we get. (4) We consider: Does it address the problem? Does it make sense when translated back into the real-world? Are the results practical, the answers reasonable, the consequences acceptable? (5) We iterate the process as needed to refine and extend our model. (6) For the real-world, practical applications, we report our results to others and implement the solution.⁶

Choices, assumptions, and approximations are present throughout the modeling cycle. It is not linear in nature, as the iterative process may have students return to a previous stage before going on to complete the modeling cycle. Functions, ratios and proportions, expressions and equations, descriptive and inferential statistical methods, and geometric representations are all important tools for analyzing mathematical modeling problems. Dynamic graphing applications, spreadsheets, simulation applications, computer algebra systems, interactive applets, and dynamic geometry software are powerful tools that can be used within the mathematical modeling process.

Mathematical modeling is best completed and interpreted not as a collection of isolated topics but integrated into other content standards. Making mathematical models is the Fourth Standard for Mathematical Practice, and specific modeling standards appear throughout the high school content standards indicated by an *(M)* symbol. The *(M)* symbol sometimes appears on the heading for a group of standards; in that case, it should be understood to apply to all standards in that group.

Mathematics | High School—Geometry

An understanding of the attributes and relationships of geometric objects can be applied in diverse contexts—interpreting a schematic drawing, estimating the amount of wood needed to frame a sloping roof, rendering computer graphics, or designing a sewing pattern for the most efficient use of material.

Although there are many types of geometry, school mathematics is devoted primarily to plane Euclidean geometry, studied both synthetically (without coordinates) and analytically (with coordinates). Euclidean geometry is characterized most importantly by the Parallel Postulate, that through a point not on a given line there is exactly one parallel line. (Spherical geometry, in contrast, has no parallel lines.)

During high school, students begin to formalize their geometry experiences from elementary and middle school, using more precise definitions and developing careful proofs. Later in college, some students develop Euclidean and other geometries carefully from a small set of axioms.

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally). Reflections and rotations each explain a particular type of symmetry, and the symmetries of an object offer insight into its attributes—as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

In the approach taken here, two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are congruent. Once these triangle congruence criteria (ASA, SAS, and SSS) are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of "same shape" and "scale factor" developed in the middle grades. These transformations lead to the criterion for triangle similarity that two pairs of corresponding angles are congruent.

The definitions of sine, cosine, and tangent for acute angles are founded on right triangles and similarity, and, with the Pythagorean Theorem, are fundamental in many real-world and theoretical situations. The Pythagorean Theorem is generalized to non-right triangles by the Law of Cosines. Together, the Laws of Sines and Cosines embody the triangle congruence criteria for the cases where three pieces of information suffice to completely solve a triangle. Furthermore, these laws yield two possible solutions in the ambiguous case, illustrating that Side-Side-Angle is not a congruence criterion.

Analytic geometry connects algebra and geometry, resulting in powerful methods of analysis and problem solving. Just as the number line associates numbers with locations in one dimension, a pair of perpendicular axes associates pairs of numbers with locations in two dimensions. This correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof. Geometric transformations of the graphs of equations correspond to algebraic changes in their equations.

Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena.

Connections to Equations.

The correspondence between numerical coordinates and geometric points allows methods from algebra to be applied to geometry and vice versa. The solution set of an equation becomes a geometric curve, making visualization a tool for doing and understanding algebra. Geometric shapes can be described by equations, making algebraic manipulation into a tool for geometric understanding, modeling, and proof.

Geometry Overview

Congruence

- Experiment with transformations in the plane.
- Understand congruence in terms of rigid motions.
- Prove geometric theorems.
- Make geometric constructions.

Similarity, Right Triangles, and Trigonometry

- Understand similarity in terms of similarity transformations.
- Prove theorems involving similarity.
- Define trigonometric ratios and solve problems involving right triangles.
- Apply trigonometry to general triangles.

Circles

- Understand and apply theorems about circles.
- Find arc lengths and areas of sectors of circles.

Expressing Geometric Properties with Equations

- Translate between the geometric description and the equation for a conic section.
- Use coordinates to prove simple geometric theorems algebraically.

Geometric Measurement and Dimension

- Explain volume formulas and use them to solve problems.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

- Visualize relationships between two-dimensional and three-dimensional objects.

Modeling with Geometry

- Apply geometric concepts in modeling situations.

Geometry

Congruence (G-CO)

Cluster	Notation	Standard
A. Experiment with transformations in the plane.	M.G.CO.A.1	1. Know precise definitions of angle, circle, perpendicular line, parallel line, and line segment, based on the undefined notions of point, line, distance along a line, and distance around a circular arc.
	M.G.CO.A.2	2. Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).
	M.G.CO.A.3	3. Given a rectangle, parallelogram, trapezoid, or regular polygon, describe the rotations and reflections that carry it onto itself.
	M.G.CO.A.4	4. Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.
	M.G.CO.A.5	5. Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

B. Understand congruence in terms of rigid motion.	M.G.CO.B.6	6. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.
	M.G.CO.B.7	7. Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.
	M.G.CO.B.8	8. Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.
C. Prove geometric theorems.	M.G.CO.C.9	9. Prove theorems about lines and angles. <i>Theorems should include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.</i>
	M.G.CO.C.10	10. Prove theorems about triangles. <i>Theorems should include: measures of interior angles of a triangle sum to 180°; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.</i>
	M.G.CO.C.11	11. Prove theorems about parallelograms. <i>Theorems should include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.</i>

Similarity, Right Triangles, and Trigonometry (G-SRT)

Cluster	Notation	Standard
A. Understand similarity in terms of similarity transformations.	M.G.SRT.A.1	Verify experimentally the properties of dilations given by a center and a scale factor: <ol style="list-style-type: none"> a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged. b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.
	M.G.SRT.A.2	Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
	M.G.SRT.A.3	Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.
B. Prove theorems involving similarity.	M.G.SRT.B.4	Prove theorems about triangles. <i>Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.</i>
	M.G.SRT.B.5	Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.
C. Define trigonometric	M.G.SRT.C.6	Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

ratios and solve problems involving right triangles.	M.G.SRT.C.7	Explain and use the relationship between the sine and cosine of complementary angles.
	M.G.SRT.C.8	Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems. (M)
D. Apply trigonometry to general triangles.	M.G.SRT.D.9	(+) Derive the formula $A = \frac{1}{2} ab \sin(C)$ for the area of a triangle by drawing an auxiliary line from a vertex perpendicular to the opposite side.
	M.G.SRT.D.10	(+) Prove the Laws of Sines and Cosines and use them to solve problems.
	M.G.SRT.D.11	(+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

Circles (G-C)

Cluster	Notation	Standard
A. Understand and apply theorems about circles.	M.G.C.A.1	Identify and describe relationships among inscribed angles, radii, and chords. <i>Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.</i>
	M.G.C.A.2	Prove properties of angles for a quadrilateral inscribed in a circle.
B. Find arc lengths and areas of sectors of circles.	M.G.C.B.3 [WI.2010. G.C.B.5]	Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Expressing Geometric Properties (G-GPE)

Cluster	Notation	Standard
A. Translate between the geometric description and the equation for a conic section.	M.G.GPE.A.1	Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
	M.G.GPE.A.2	Derive the equation of a parabola given a focus and directrix.
	M.G.GPE.A.3	(+) Derive the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from the foci is constant.
B. Use coordinates to prove simple geometric theorems algebraically.	M.G.GPE.B.4	Use coordinates to prove simple geometric theorems algebraically. <i>For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point $(1, \sqrt{3})$ lies on the circle centered at the origin and containing the point $(0, 2)$.</i>
	M.G.GPE.B.5	Prove the slope criteria for parallel and perpendicular lines and use them to solve geometric problems (e.g., find the equation of a line parallel or perpendicular to a given line that passes through a given point).
	M.G.GPE.B.6	Find the point on a directed line segment between two given points that partitions the segment in a given ratio.
	M.G.GPE.B.7	Use coordinates to compute perimeters of polygons and areas of triangles and rectangles (e.g., using the distance formula).

Geometric Measurement and Dimension (G-GMD)

Cluster	Notation	Standard
A. Explain volume formulas and use them to solve problems.	M.G.GMD.A.1	Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone.
	M.G.GMD.A.2 [WI.2010. G.GMC.A.3]	Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems. (M)
B. Visualize relationships between two-dimensional and three-dimensional objects.	M.G.GMD.B.3 [WI.2010. G.GMD.4]	Identify three-dimensional objects generated by rotations of two-dimensional objects.

Modeling with Geometry (G-MG)

Cluster Statement	Notation	Standard
A. Apply geometric concepts in modeling situations.	M.G.MG.A.1	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder). (M)
	M.G.MG.A.2	Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot). (M)
	M.G.MG.A.3	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). (M)

Mathematics - High School Statistics & Probability: Introduction

Decisions or predictions are often based on data—numbers in context. These decisions or predictions would be easy if the data always sent a clear message, but the message is often obscured by variability. Statistics provides tools for describing variability in data and for making informed decisions that take it into account.

Data are gathered, displayed, summarized, examined, and interpreted to discover patterns and deviations from patterns. Quantitative data can be described in terms of key characteristics: measures of shape, center, and spread. The shape of a data distribution might be described as symmetric, skewed, flat, or bell shaped, and it might be summarized by a statistic measuring center (such as mean or median) and a statistic measuring spread (such as standard deviation or interquartile range). Different distributions can be compared numerically using these statistics or compared visually using plots. Knowledge of center and spread are not enough to describe a distribution. Which statistics to compare, which plots to use, and what the results of a comparison might mean, depend on the question to be investigated and the real-life actions to be taken.

Throughout the statistics and probability standards, students build on the work they did in 6th and 8th grades working with descriptive statistics and linear modeling. They also build on the development of probability in 7th grade as they work with independent and dependent events and conditional probabilities.

Students further develop their understanding of inferential statistics using simulation, which is grounded in their work from 7th grade. Students also use the probability tools they developed during 7th grade and in the foundational standards to make decisions about payoffs for a game or analyzing the testing of a product.

Randomization has two important uses in drawing statistical conclusions. First, collecting data from a random sample of a population makes it possible to draw valid conclusions about the whole population, taking variability into account. Second, randomly assigning individuals to different treatments allows a fair comparison of the effectiveness of those treatments. A statistically significant outcome is one that is unlikely to be due to chance alone, and this can be evaluated only under the condition of randomness. The conditions under which data are collected are important in drawing conclusions from the data; in critically reviewing uses of statistics in public media and other reports, it is important to consider the study design, how the data were gathered, and the analyses employed as well as the data summaries and the conclusions drawn.

Random processes can be described mathematically by using a probability model: a list or description of the possible outcomes (the sample space), each of which is assigned a probability. In situations such as flipping a coin, rolling a number cube, or drawing

a card, it might be reasonable to assume various outcomes are equally likely. In a probability model, sample points represent outcomes and combine to make up events; probabilities of events can be computed by applying the Addition and Multiplication Rules. Interpreting these probabilities relies on an understanding of independence and conditional probability, which can be approached through the analysis of two-way tables.

Technology plays an important role in statistics and probability by making it possible to generate plots, regression functions, and correlation coefficients, and to simulate many possible outcomes in a short amount of time.

Connections to Functions and Modeling.

Functions may be used to describe data; if the data suggest a linear relationship, the relationship can be modeled with a regression line, and its strength and direction can be expressed through a correlation coefficient.

Statistics and Probability Overview

Interpreting Categorical and Quantitative Data

- Summarize, represent, and interpret data on a single count or measurement variable.
- Summarize, represent, and interpret data on two categorical and quantitative variables.
- Interpret linear models.

Making Inferences and Justifying Conclusions

- Understand and evaluate random processes underlying statistical experiments.
- Make inferences and justify conclusions from sample surveys, experiments, and observational studies.

Conditional Probability and the Rules of Probability

- Understand independence and conditional probability and use them to interpret data.
- Use the rules of probability to compute probabilities of compound events in a uniform probability model.

Using Probability to Make Decisions

- Calculate expected values and use them to solve problems.
- Use probability to evaluate outcomes of decisions.

Mathematical Practices

Make sense of problems and persevere in solving them.

Reason abstractly and quantitatively.

Construct viable arguments, and appreciate and critique the reasoning of others.

Model with mathematics.

Use appropriate tools strategically.

Attend to precision.

Look for and make use of structure.

Look for and express regularity in repeated reasoning.

Statistics and Probability (SP) (M)

Interpreting Categorical and Quantitative Data (S-ID)

Cluster Statement	Notation	Standard
A. Summarize, represent, and interpret data on a single count or measurement variable.	M.SP.ID.A.1	Represent data with plots on the real number line (dot plots, histograms, and box plots).
	M.SP.ID.A.2	Use statistics appropriate to the shape of the data distribution to compare center (median, mean) and spread (interquartile range, standard deviation) of two or more different data sets.
	M.SP.ID.A.3	Interpret differences in shape, center, and spread in the context of the data sets, accounting for possible effects of extreme data points (outliers).
	M.SP.ID.A.4	Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.
B. Summarize, represent, and interpret data on two categorical and quantitative variables.	M.SP.ID.B.5	Summarize categorical data for two categories in two-way frequency tables. Interpret relative frequencies in the context of the data (including joint, marginal, and conditional relative frequencies as examples of proportionality and disproportionality). Recognize possible associations and trends in the data.
	M.SP.ID.B.6	Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of

		<p>the data. <i>Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.</i></p> <p>b. Informally assess the fit of a function by plotting and analyzing residuals.</p> <p>c. Fit a linear function for a scatter plot that suggests a linear association.</p>
C. Interpret linear models	M.SP.ID.C.7	Interpret the slope (rate of change) and the intercept (constant term) of a linear model in the context of the data.
	M.SP.ID.C.8	Compute (using technology) and interpret the correlation coefficient of a linear fit.
	M.SP.ID.C.9	Distinguish between correlation and causation.

Making Inferences and Justifying Conclusions (S-IC)

Cluster Statement	Notation	Standard
A. Understand and evaluate random processes underlying statistical experiments.	M.SP.IC.A.1	Understand statistics as a process for making inferences about population parameters based on a random sample from that population.
	M.SP.IC.A.2	Decide if a specified model is consistent with results from a given data-generating process (e.g., using simulation). <i>For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?</i>
B. Make inferences and justify conclusions from sample surveys, experiments, and observational studies.	M.SP.IC.B.3	Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each.
	M.SP.IC.B.4	Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.
	M.SP.IC.B.5	Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.
	M.SP.IC.B.6	Evaluate reports based on data.

Conditional Probability and the Rules of Probability (S-CP)

Cluster Statement	Notation	Standard
A. Understand independence and conditional probability and use them to interpret data.	M.SP.CP.A.1	Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").
	M.SP.CP.A.2	Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.
	M.SP.CP.A.3	Understand the conditional probability of A given B as $P(A \text{ and } B)/P(B)$, and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A , and the conditional probability of B given A is the same as the probability of B .
	M.SP.CP.A.4	Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. <i>For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.</i>
	M.SP.CP.A.5	Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. <i>For example, compare the chance of having lung cancer if you are a smoker with the chance of being a</i>

		<i>smoker if you have lung cancer.</i>
B. Use the rules of probability to compute probabilities of compound events in a uniform probability model.	M.SP.CP.B.6	Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.
	M.SP.CP.B.7	Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.
	M.SP.CP.B.8	(+) Apply the general Multiplication Rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B A) = P(B)P(A B)$, and interpret the answer in terms of the model.
	M.SP.CP.B.9	(+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Using Probability to Make Decisions (S-MD)

Cluster Statement	Notation	Standard
A. Calculate expected values and use them to solve problems.	M.SP.MD.A.1	(+) Define a random variable for a quantity of interest by assigning a numerical value to each event in a sample space; graph the corresponding probability distribution using the same graphical displays as for data distributions.
	M.SP.MD.A.2	(+) Calculate the expected value of a random variable; interpret it as the mean of the probability distribution.
	M.SP.MD.A.3	(+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value. <i>For example, find the theoretical probability distribution for the number of correct answers obtained by guessing on all five questions of a multiple-choice test where each question has four choices, and find the expected grade under various grading schemes.</i>
	M.SP.MD.A.4	(+) Develop a probability distribution for a random variable defined for a sample space in which probabilities are assigned empirically; find the expected value. <i>For example, find a current data distribution on the number of TV sets per household in the United States, and calculate the expected number of sets per household. How many TV sets would you expect to find in 100 randomly selected households?</i>
B. Use probability to evaluate outcomes of	M.SP.MD.B.5	(+) Weigh the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values. a. Find the expected payoff for a game of chance.

decisions.		<p><i>For example, find the expected winnings from a state lottery ticket or a game at a fast-food restaurant.</i></p> <p>b. Evaluate and compare strategies on the basis of expected values.</p> <p><i>For example, compare a high-deductible versus a low-deductible automobile insurance policy using various, but reasonable, chances of having a minor or a major accident.</i></p>
	M.SP.MD.B.6	(+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).
	M.SP.MD.B.7	(+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

Endnotes

1. Gutierrez, Rochelle. (2017). Political conocimiento for teaching mathematics: Why teachers need it and how to develop it.
2. Appendix II: Standards for Mathematical Practice Grade-Span Descriptions: PreK-5, 6-8, 9-12. *Massachusetts Curriculum Framework for Mathematics*. 2017. 171-179. Retrieved from <https://www.doe.mass.edu/frameworks/math/2017-06.pdf>
3. Student Achievement Partners. 2014. *Mathematics: Focus By Grade Level*. Retrieved from <https://achievethecore.org/index.php/category/774/mathematics-focus-by-grade-level>
4. *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)*. 2016. United States of America; Consortium for Mathematics and Its Applications (COMAP, Inc.) and Society for Industrial and Applied Mathematics (SIAM). 32.
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